Functions and Their Graphs

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Contents

1 Functions

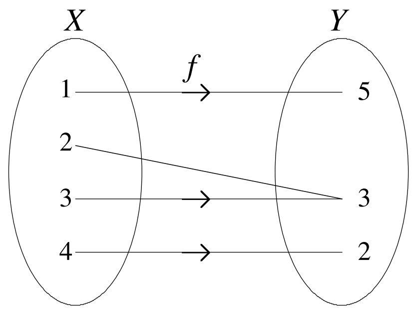
In this Chapter we will cover various aspects of functions. We will look at the definition of a function, the domain and range of a function, what we mean by specifying the domain of a function and absolute value function.

1.1 What is a function?

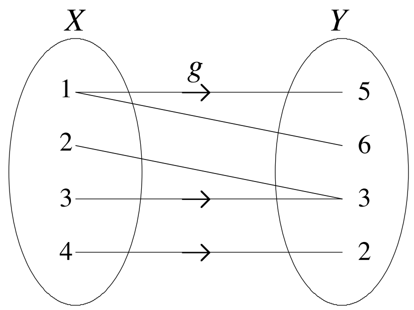
1.1.1 Definition of a function

A function f from a set of elements X to a set of elements Y is a rule that assigns to each element x in X exactly one element y in Y.

One way to demonstrate the meaning of this definition is by using arrow diagrams.



$f : X \to Y$ is a function. Every element in X has associated with it exactly one element of Y.



$g : X \to Y$ is not a function. The element 1 in set X is assigned two elements, 5 and 6 in set Y.

A function can also be described as a set of ordered pairs (x, y) such that for any x-value in the set, there is only one y-value. This means that there cannot be any repeated x-values with different y-values.

The examples above can be described by the following sets of ordered pairs.

F = {(1,5),(3,3),(2,3),(4,2)} is a function.

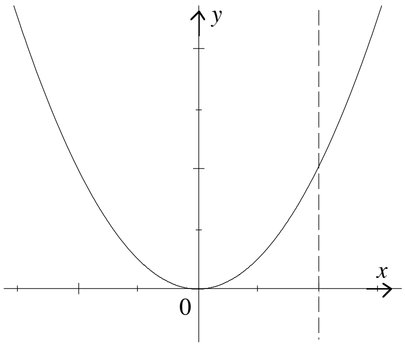
G = {(1,5),(4,2),(2,3),(3,3),(1,6)} is not a function.

The definition we have given is a general one. While in the examples we have used numbers as elements of X and Y, there is no reason why this must be so. However, in these notes we will only consider functions where X and Y are subsets of the real numbers.

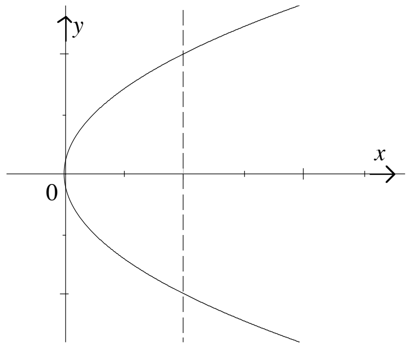
In this setting, we often describe a function using the rule, $y = f(x)$, and create a graph of that function by plotting the ordered pairs $(x, f(x))$ on the Cartesian Plane. This graphical representation allows us to use a test to decide whether or not we have the graph of a function: The Vertical Line Test.

1.1.2 The Vertical Line Test

The Vertical Line Test states that if it is *not possible* to draw a vertical line through a graph so that it cuts the graph in more than one point, then the graph *is* a function.



This is the graph of a function. All possible vertical lines will cut this graph only once.



This is not the graph of a function. The vertical line we have drawn cuts the graph twice.

1.1.3 Domain of a function

For a function $f : X \to Y$ the *domain* of f is the set X.

This also corresponds to the set of x-values when we describe a function as a set of ordered pairs (x, y).

If only the rule $y = f(x)$ is given, then the domain is taken to be the set of all real x for which the function is defined. For example, $y = \sqrt{x}$ has domain; all real $x \geq 0$. This is sometimes referred to as the *natural* domain of the function.

1.1.4 Range of a function

For a function $f : X \to Y$ the *range* of f is the set of y-values such that $y = f(x)$ for some x in X.

This corresponds to the set of y-values when we describe a function as a set of ordered pairs (x, y). The function $y = \sqrt{x}$ has range; all real $y \geq 0$.

Example

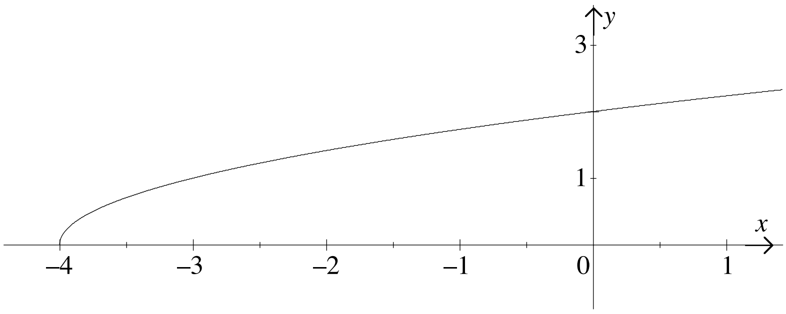
a. State the domain and range of $y = \sqrt{x+4}$.

b. Sketch, showing significant features, the graph of $y = \sqrt{x+4}$.

Solution

a. The domain of $y = \sqrt{x+4}$ is all real $x \geq -4$. We know that square root functions are only defined for positive numbers so we require that $x + 4 \geq 0$, ie $x \geq -4$. We also know that the square root functions are always positive so the range of $y = \sqrt{x+4}$ is all real $y \geq 0$.

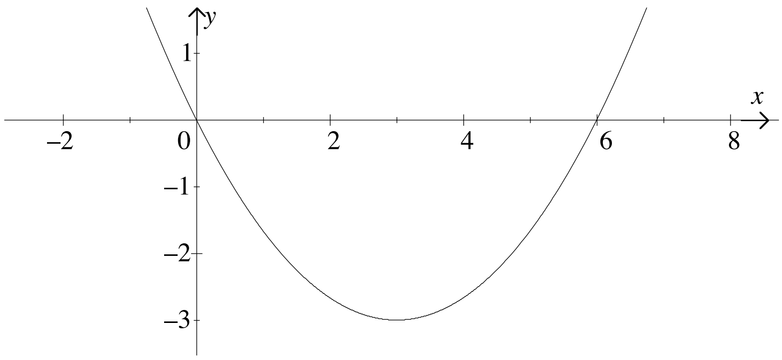
b.



The graph of $y = \sqrt{x+4}$.

Example

a. State the equation of the parabola sketched below, which has vertex $(3, -3)$.



b. Find the domain and range of this function.

Solution

a. The equation of the parabola is $y = \frac{x^2-6x}{3}$.

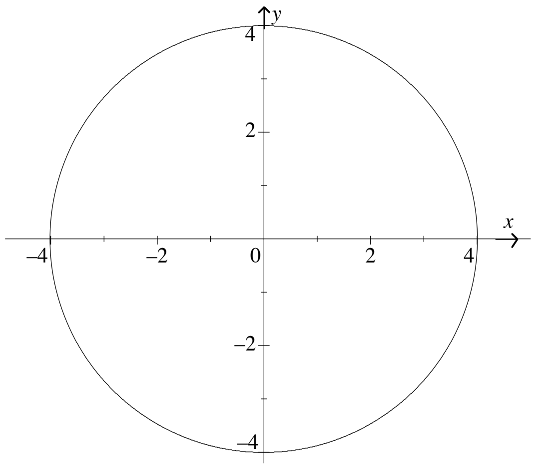
b. The domain of this parabola is all real x. The range is all real $y \geq -3$.

Example

Sketch $x^2 + y^2 = 16$ and explain why it is not the graph of a function.

Solution

$x^2 + y^2 = 16$ is not a function as it fails the vertical line test. For example, when $x = 0$ $y = \pm 4$.



The graph of $x^2 + y^2 = 16$.

Example

Sketch the graph of $f(x) = 3x - x^2$ and find

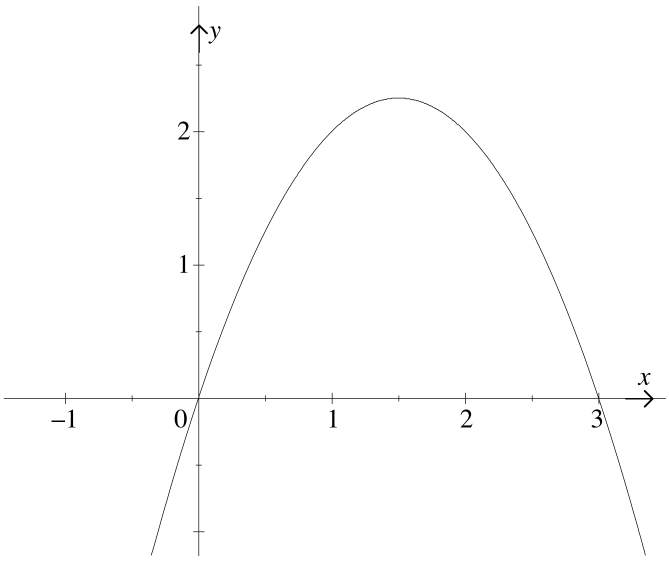
a. the domain and range

b. $f(q)$

c. $f(x^2)$

d. $\frac{f(2+h)-f(2)}{h}$, $h \neq 0$.

Solution



The graph of $f(x) = 3x - x^2$.

a. The domain is all real x. The range is all real y where $y \leq 2.25$.

b. $f(q) = 3q - q^2$

c. $f(x^2) = 3(x^2) - (x^2)^2 = 3x^2 - x^4$

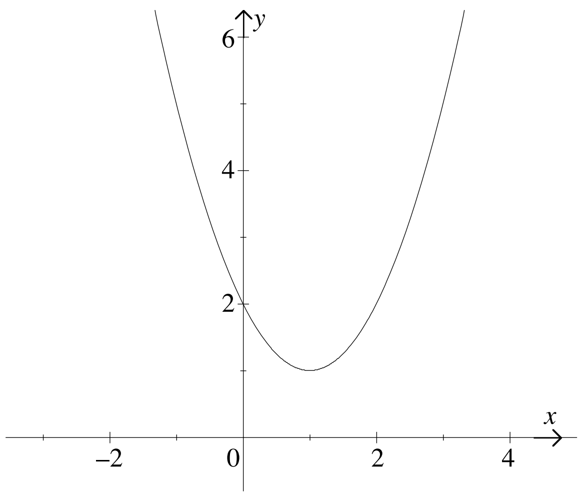
d.

$$\begin{aligned}\frac{f(2+h)-f(2)}{h} &= \frac{(3(2+h)-(2+h)^2)-(3(2)-(2)^2)}{h}\\ &= \frac{6+3h-(h^2+4h+4)-2}{h}\\ &= \frac{-h^2-h}{h}\\ &= -h-1\end{aligned}$$

Example

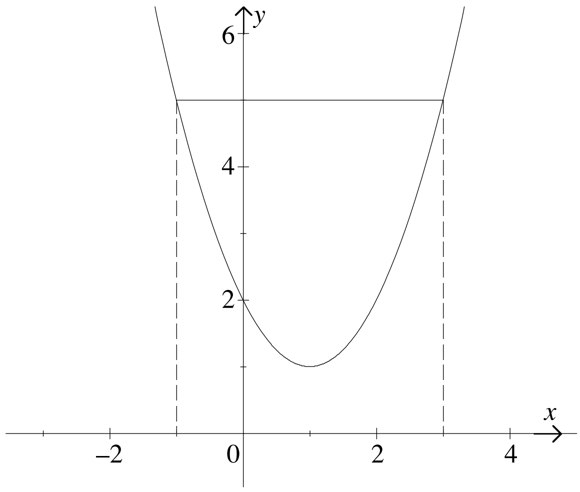
Sketch the graph of the function $f(x) = (x-1)^2 + 1$ and show that $f(p) = f(2-p)$. Illustrate this result on your graph by choosing one value of p.

Solution



The graph of $f(x) = (x-1)^2 + 1$.

$$\begin{aligned}f(2-p) &= ((2-p)-1)^2+1\\ &= (1-p)^2+1\\ &= (p-1)^2+1\\ &= f(p)\end{aligned}$$



The sketch illustrates the relationship $f(p) = f(2-p)$ for $p = -1$. If $p = -1$ then $2 - p = 2 - (-1) = 3$, and $f(-1) = f(3)$.

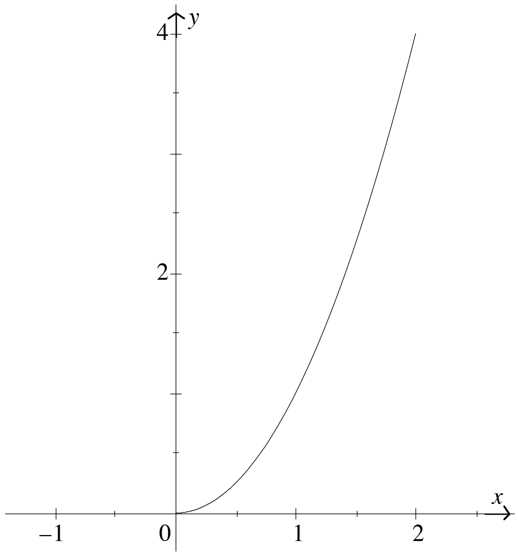
1.2 Specifying or restricting the domain of a function

We sometimes give the rule $y = f(x)$ along with the domain of definition. This domain may not necessarily be the natural domain. For example, if we have the function

$$y = x^2 \quad \text{for} \quad 0 \leq x \leq 2$$

then the domain is given as $0 \leq x \leq 2$. The natural domain has been restricted to the subinterval $0 \leq x \leq 2$.

Consequently, the range of this function is all real y where $0 \leq y \leq 4$. We can best illustrate this by sketching the graph.



The graph of $y = x^2$ for $0 \leq x \leq 2$.

1.3 The absolute value function

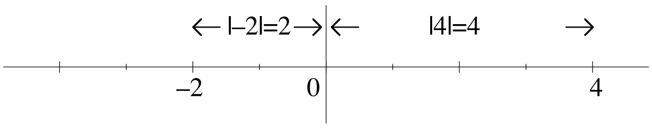
Before we define the absolute value function we will review the definition of the absolute value of a number.

The *Absolute value of a number* x is written $|x|$ and is defined as

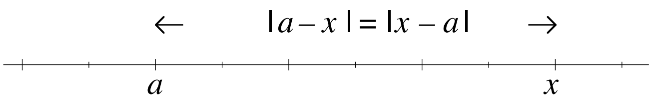
$$|x| = x \text{ if } x \geq 0 \qquad \text{or} \qquad |x| = -x \text{ if } x < 0.$$

That is, $|4| = 4$ since 4 is positive, but $|-2| = 2$ since -2 is negative.

We can also think of $|x|$ geometrically as the distance of x from 0 on the number line.



More generally, $|x - a|$ can be thought of as the distance of x from a on the numberline.



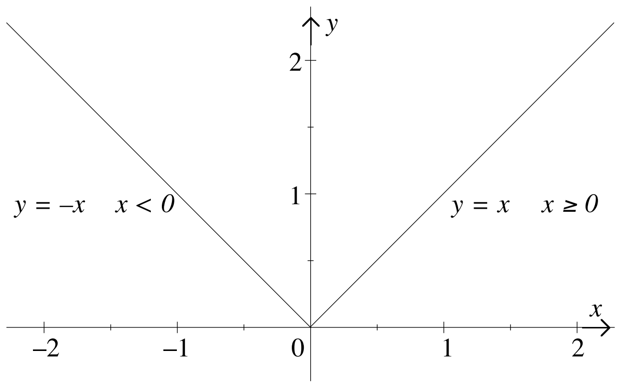
Note that $|a - x| = |x - a|$.

The absolute value *function* is written as $y = |x|$.

We define this function as

$$y = \begin{cases} +x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

From this definition we can graph the function by taking each part separately. The graph of $y = |x|$ is given below.



The graph of $y = |x|$.

Example

Sketch the graph of $y = |x - 2|$.

Solution

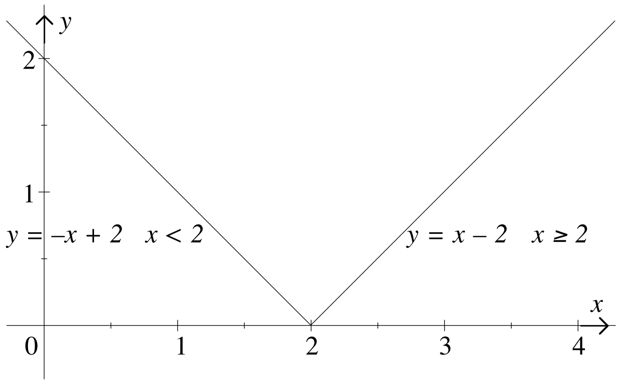
For $y = |x - 2|$ we have

$$y = \begin{cases} +(x-2) & \text{when } x - 2 \geq 0 \quad \text{or} \quad x \geq 2 \\ -(x-2) & \text{when } x - 2 < 0 \quad \text{or} \quad x < 2 \end{cases}$$

That is,

$$y = \begin{cases} x - 2 & \text{for } x \geq 2 \\ -x + 2 & \text{for } x < 2 \end{cases}$$

Hence we can draw the graph in two parts.



The graph of $y = |x - 2|$.

We could have sketched this graph by first of all sketching the graph of $y = x - 2$ and then reflecting the negative part in the x-axis. We will use this fact to sketch graphs of this type in Chapter 2.

1.4 Exercises

1. **a.** State the domain and range of $f(x) = \sqrt{9 - x^2}$.

b. Sketch the graph of $y = \sqrt{9 - x^2}$.

2. Given $\psi(x) = x^2 + 5$, find, in simplest form, $\dfrac{\psi(x+h) - \psi(x)}{h}$ $\quad h \neq 0$.

3. Sketch the following functions stating the domain and range of each:

a. $y = \sqrt{x-1}$

b. $y = |2x|$

c. $y = \frac{1}{x-4}$

d. $y = |2x| - 1$.

4. **a.** Find the perpendicular distance from $(0, 0)$ to the line $x + y + k = 0$

b. If the line $x + y + k = 0$ cuts the circle $x^2 + y^2 = 4$ in two distinct points, find the restrictions on k.

5. Sketch the following, showing their important features.

a. $y = \left(\frac{1}{2}\right)^x$

b. $y^2 = x^2$.

6. Explain the meanings of function, domain and range. Discuss whether or not $y^2 = x^3$ is a function.

7. Sketch the following relations, showing all intercepts and features. State which ones are functions giving their domain and range.

a. $y = -\sqrt{4 - x^2}$

b. $|x| - |y| = 0$

c. $y = x^3$

d. $y = \frac{x}{|x|}, x \neq 0$

e. $|y| = x$.

8. If $A(x) = x^2 + 2 + \frac{1}{x^2}$, $x \neq 0$, prove that $A(p) = A(\frac{1}{p})$ for all $p \neq 0$.

9. Write down the values of x which are not in the domain of the following functions:

a. $f(x) = \sqrt{x^2 - 4x}$

b. $g(x) = \frac{x}{x^2-1}$

10. If $\phi(x) = \log\left(\frac{x}{x-1}\right)$, find in simplest form:

a. $\phi(3) + \phi(4) + \phi(5)$

b. $\phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n)$

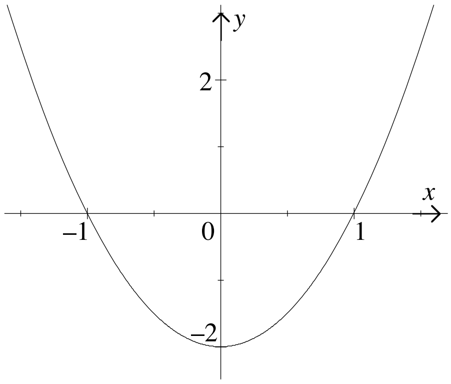
11. **a.** If $y = x^2 + 2x$ and $x = (z - 2)^2$, find y when $z = 3$.

b. Given $L(x) = 2x + 1$ and $M(x) = x^2 - x$, find

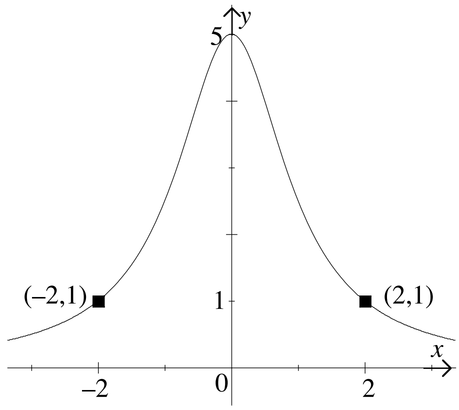
i. $L(M(x))$

ii. $M(L(x))$

12. Using the sketches, find the value(s) of the constants in the given equations:



$y = ax^2 - b$



$y = \frac{a}{bx^2+1}$

13. a. Define $|a|$, the absolute value of a, where a is real.

b. Sketch the relation $|x| + |y| = 1$.

14. Given that $S(n) = \frac{n}{2n+1}$, find an expression for $S(n-1)$.

Hence show that $S(n) - S(n-1) = \frac{1}{(2n-1)(2n+1)}$.

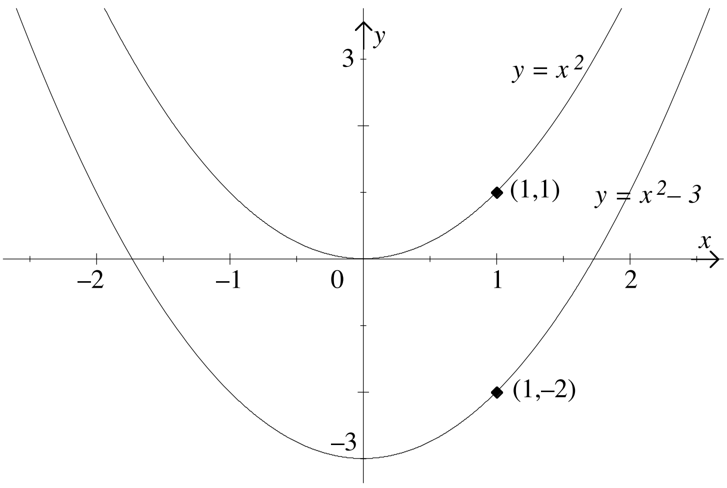
2 More about functions

In this Chapter we will look at the effects of stretching, shifting and reflecting the basic functions, $y = x^2$, $y = x^3$, $y = \frac{1}{x}$, $y = |x|$, $y = a^x$, $x^2 + y^2 = r^2$. We will introduce the concepts of even and odd functions, increasing and decreasing functions and will solve equations using graphs.

2.1 Modifying functions by shifting

2.1.1 Vertical shift

We can draw the graph of $y = f(x) + k$ from the graph of $y = f(x)$ as the addition of the constant k produces a *vertical shift*. That is, adding a constant to a function moves the graph up k units if $k > 0$ or down k units if $k < 0$. For example, we can sketch the function $y = x^2 - 3$ from our knowledge of $y = x^2$ by shifting the graph of $y = x^2$ *down* by 3 units. That is, if $f(x) = x^2$ then $f(x) - 3 = x^2 - 3$.

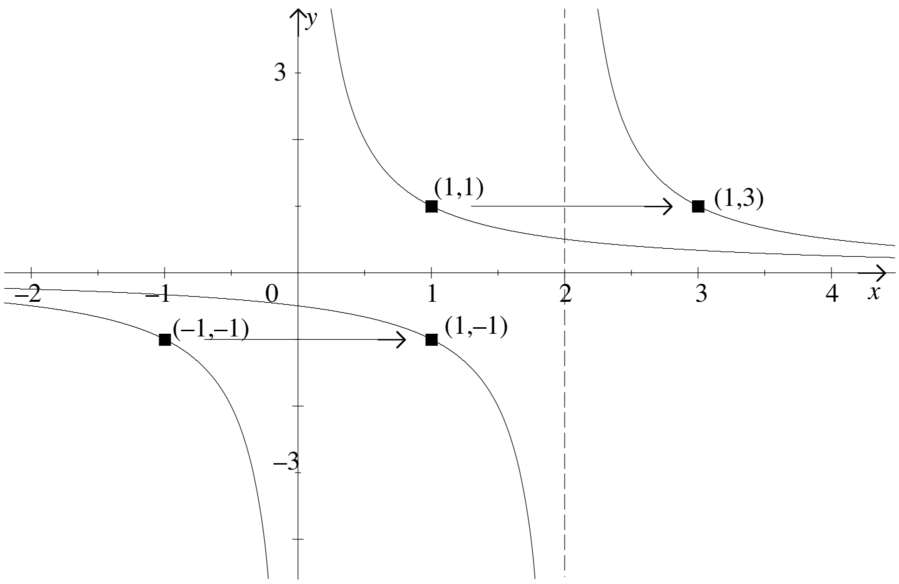


We can also write $y = f(x) - 3$ as $y + 3 = f(x)$, so replacing y by $y + 3$ in $y = f(x)$ also shifts the graph down by 3 units.

2.1.2 Horizontal shift

We can draw the graph of $y = f(x - a)$ if we know the graph of $y = f(x)$ as placing the constant a inside the brackets produces a *horizontal shift*. If we replace x by $x - a$ inside the function then the graph will shift to the left by a units if $a < 0$ and to the right by a units if $a > 0$.

For example we can sketch the graph of $y = \frac{1}{x-2}$ from our knowledge of $y = \frac{1}{x}$ by shifting this graph to the right by 2 units. That is, if $f(x) = \frac{1}{x}$ then $f(x-2) = \frac{1}{x-2}$.

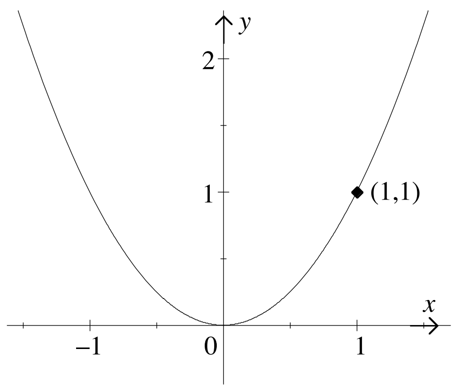


Note that the function $y = \frac{1}{x-2}$ is not defined at $x = 2$. The point $(1,1)$ has been shifted to $(1,3)$.

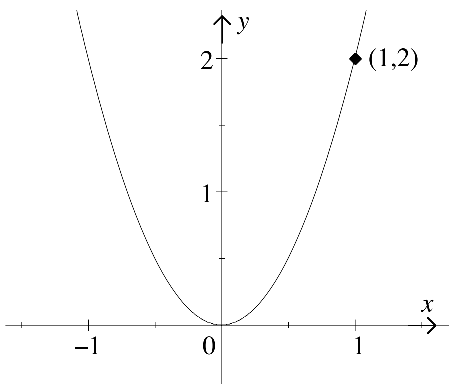
2.2 Modifying functions by stretching

We can sketch the graph of a function $y = bf(x)$ $(b > 0)$ if we know the graph of $y = f(x)$ as multiplying by the constant b will have the effect of stretching the graph in the y-direction by a factor of b. That is, multiplying $f(x)$ by b will change all of the y-values proportionally.

For example, we can sketch $y = 2x^2$ from our knowledge of $y = x^2$ as follows:

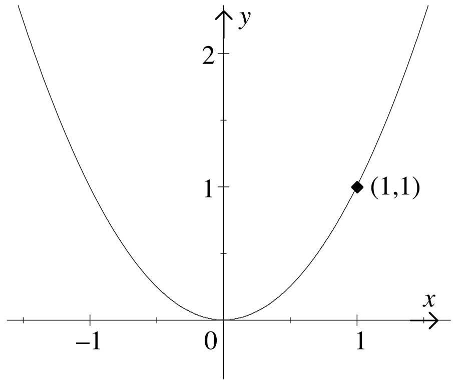


The graph of $y = x^2$.

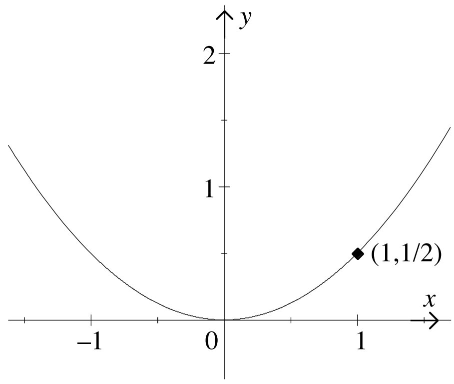


The graph of $y = 2x^2$. Note, all the y-values have been multiplied by 2, but the x-values are unchanged.

We can sketch the graph of $y = \frac{1}{2}x^2$ from our knowledge of $y = x^2$ as follows:



The graph of $y = x^2$.

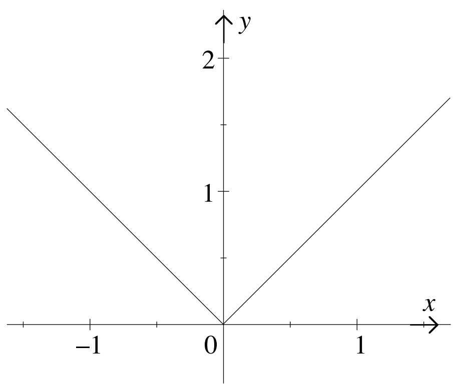


The graph of $y = \frac{1}{2}x^2$. Note, all the y-values have been multiplied by $\frac{1}{2}$, but the x-values are unchanged.

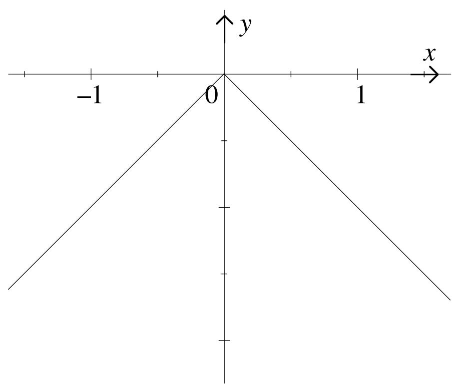
2.3 Modifying functions by reflections

2.3.1 Reflection in the x-axis

We can sketch the function $y = -f(x)$ if we know the graph of $y = f(x)$, as a minus sign in front of $f(x)$ has the effect of reflecting the whole graph in the x-axis. (Think of the x-axis as a mirror.) For example, we can sketch $y = -|x|$ from our knowledge of $y = |x|$.



The graph of $y = |x|$.

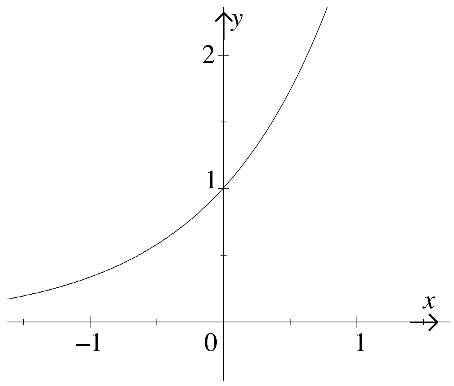


The graph of $y = -|x|$. It is the reflection of $y = |x|$ in the x-axis.

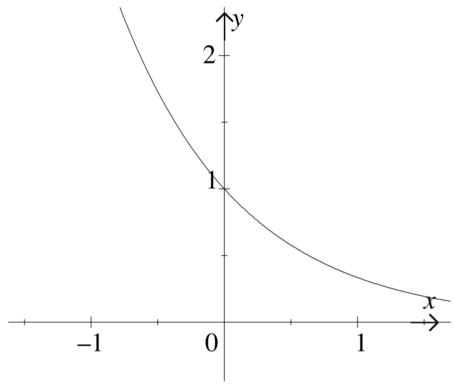
2.3.2 Reflection in the y-axis

We can sketch the graph of $y = f(-x)$ if we know the graph of $y = f(x)$ as the graph of $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.

For example, we can sketch $y = 3^{-x}$ from our knowledge of $y = 3^x$.



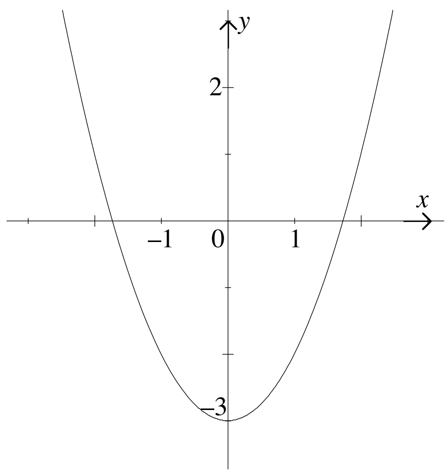
The graph of $y = 3^x$.



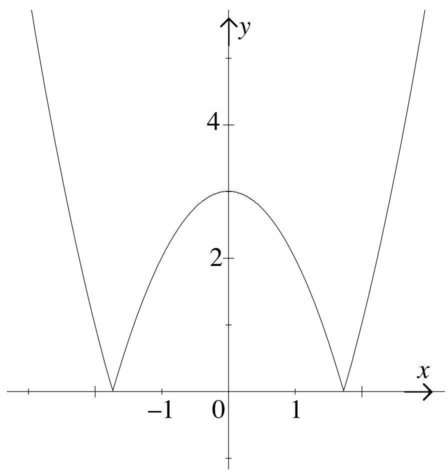
The graph of $y = 3^{-x}$. It is the reflection of $y = 3^x$ in the y-axis.

2.4 Other effects

We can sketch the graph of $y = |f(x)|$ if we know the graph of $y = f(x)$ as the effect of the absolute value is to reflect all of the *negative* values of $f(x)$ in the x-axis. For example, we can sketch the graph of $y = |x^2 - 3|$ from our knowledge of the graph of $y = x^2 - 3$.



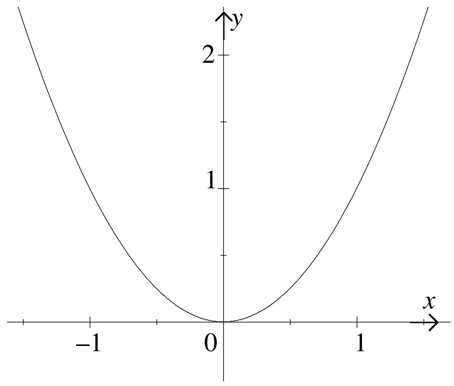
The graph of $y = x^2 - 3$.



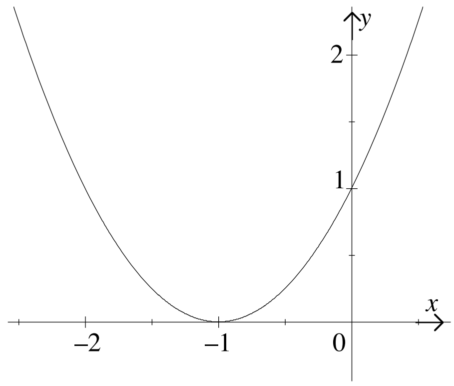
The graph of $y = |x^2 - 3|$. The negative values of $y = x^2 - 3$ have been reflected in the x-axis.

2.5 Combining effects

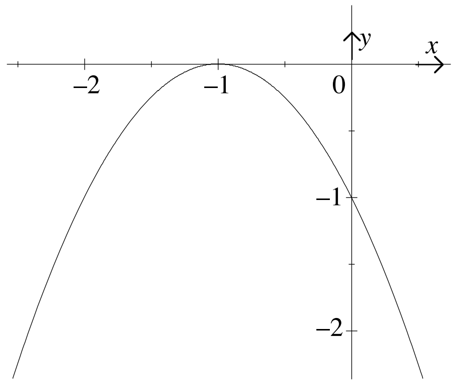
We can use all the above techniques to graph more complex functions. For example, we can sketch the graph of $y = 2-(x+1)^2$ from the graph of $y = x^2$ provided we can analyse the combined effects of the modifications. Replacing x by $x + 1$ (or $x - (-1)$) moves the graph to the left by 1 unit. The effect of the − sign in front of the brackets turns the graph up side down. The effect of adding 2 moves the graph up 2 units. We can illustrate these effects in the following diagrams.



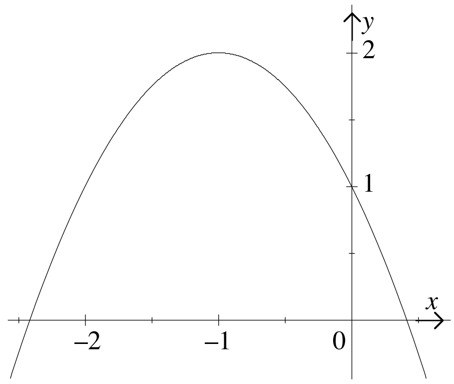
The graph of $y = x^2$.



The graph of $y = (x + 1)^2$. The graph of $y = x^2$ has been shifted 1 unit to the left.



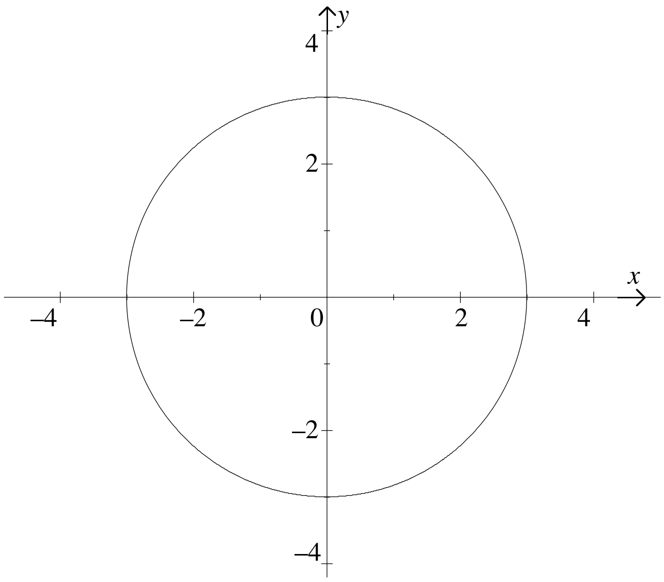
The graph of $y = -(x + 1)^2$. The graph of $y = (x + 1)^2$ has been reflected in the x-axis.



The graph of $y = 2-(x+1)^2$. The graph of $y = -(x + 1)^2$ has been shifted up by 2 units.

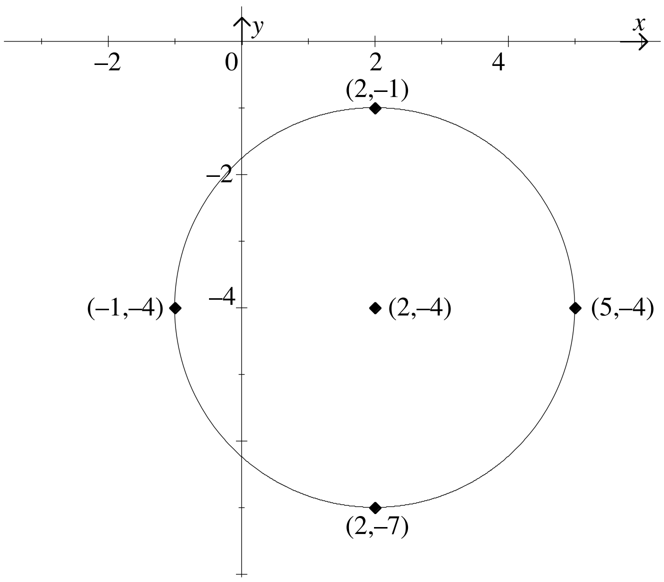
Similarly, we can sketch the graph of $(x-h)^2+(y-k)^2 = r^2$ from the graph of $x^2+y^2 = r^2$. Replacing x by $x - h$ shifts the graph sideways h units. Replacing y by $y - k$ shifts the graph up or down k units. (We remarked before that $y = f(x) + k$ could be written as $y - k = f(x)$.)

For example, we can use the graph of the circle of radius 3, $x^2 + y^2 = 9$, to sketch the graph of $(x - 2)^2 + (y + 4)^2 = 9$.



The graph of $x^2 + y^2 = 9$.

This is a circle centre $(0, 0)$, radius 3.



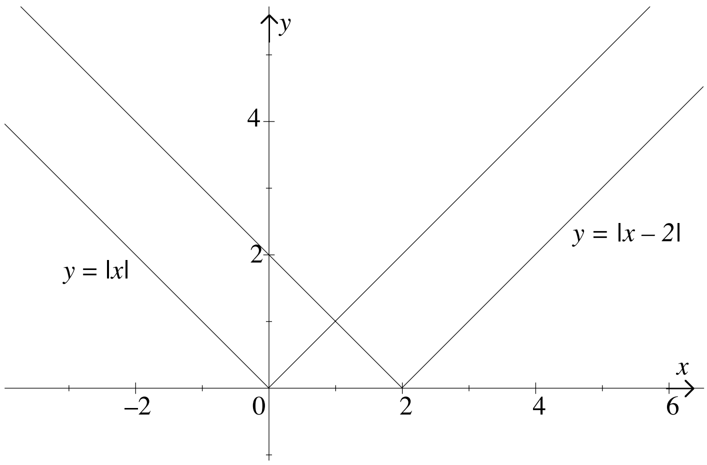
The graph of $(x - 2)^2 + (y + 4)^2 = 9$.

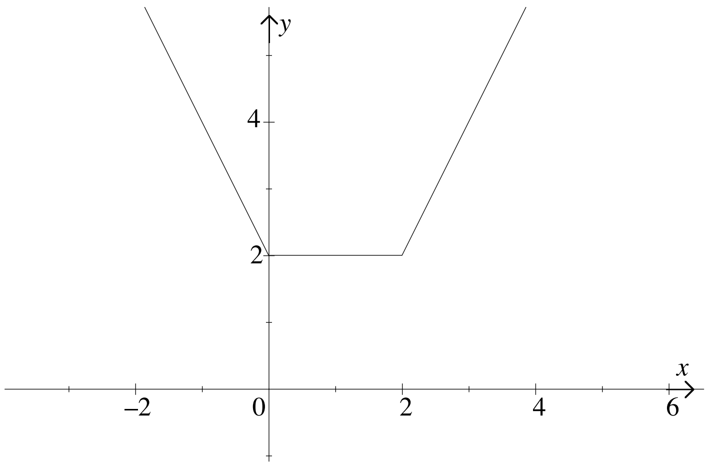
This is a circle centre $(2, -4)$, radius 3.

Replacing x by $x - 2$ has the effect of shifting the graph of $x^2 + y^2 = 9$ two units to the right. Replacing y by $y + 4$ shifts it down 4 units.

2.6 Graphing by addition of ordinates

We can sketch the graph of functions such as $y = |x| + |x - 2|$ by drawing the graphs of both $y = |x|$ and $y = |x - 2|$ on the same axes then adding the corresponding y-values.





The graph of $y = |x| + |x - 2|$.

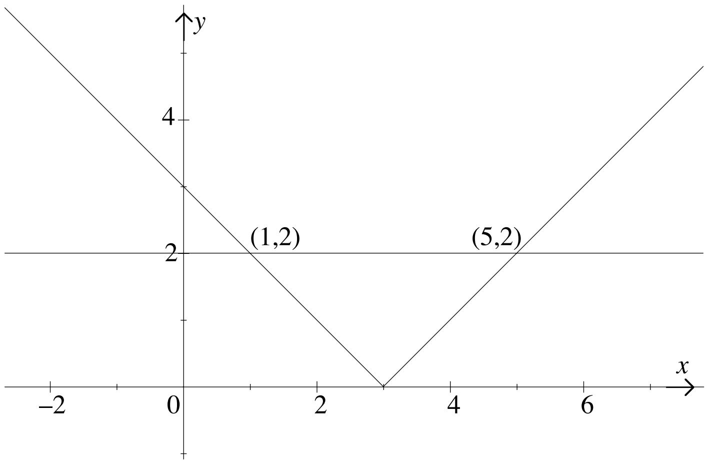
At each point of x the y-values of $y = |x|$ and $y = |x - 2|$ have been added. This allows us to sketch the graph of $y = |x| + |x - 2|$.

This technique for sketching graphs is very useful for sketching the graph of the sum of two trigonometric functions.

2.7 Using graphs to solve equations

We can solve equations of the form $f(x) = k$ by sketching $y = f(x)$ and the horizontal line $y = k$ on the same axes. The solution to the equation $f(x) = k$ is found by determining the x-values of any points of intersection of the two graphs.

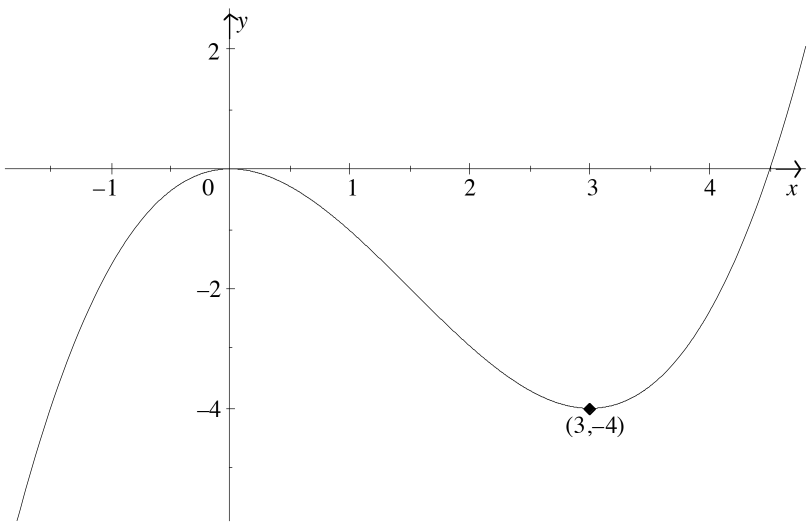
For example, to solve $|x-3|=2$ we sketch $y=|x-3|$ and $y=2$ on the same axes.



The x-values of the points of intersection are 1 and 5. Therefore $|x-3|=2$ when $x=1$ or $x=5$.

Example

The graph of $y=f(x)$ is sketched below.



For what values of k does the equation $f(x)=k$ have

1. 1 solution

2. 2 solutions

3. 3 solutions?

Solution

If we draw a horizontal line $y=k$ across the graph $y=f(x)$, it will intersect once when $k>0$ or $k<-4$, twice when $k=0$ or $k=-4$ and three times when $-4<k<0$. Therefore the equation $f(x)=k$ will have

1. 1 solution if $k > 0$ or $k < -4$

2. 2 solutions if $k = 0$ or $k = -4$

3. 3 solutions if $-4 < k < 0$.

2.8 Exercises

1. Sketch the following:

a. $y = x^2$ **b.** $y = \frac{1}{3}x^2$ **c.** $y = -x^2$ **d.** $y = (x+1)^2$

2. Sketch the following:

a. $y = \frac{1}{x}$ **b.** $y = \frac{1}{x-2}$ **c.** $y = \frac{-2}{x}$ **d.** $y = \frac{1}{x+1} + 2$

3. Sketch the following:

a. $y = x^3$ **b.** $y = |x^3 - 2|$ **c.** $y = 3 - (x-1)^3$

4. Sketch the following:

a. $y = |x|$ **b.** $y = 2|x-2|$ **c.** $y = 4 - |x|$

5. Sketch the following:

a. $x^2 + y^2 = 16$ **b.** $x^2 + (y+2)^2 = 16$ **c.** $(x-1)^2 + (y-3)^2 = 16$

6. Sketch the following:

a. $y = \sqrt{9 - x^2}$ **b.** $y = \sqrt{9 - (x-1)^2}$ **c.** $y = \sqrt{9 - x^2} - 3$

7. Show that $\dfrac{x-1}{x-2} = \dfrac{1}{x-2} + 1$.

Hence sketch the graph of $y = \dfrac{x-1}{x-2}$.

8. Sketch $y = \frac{x+1}{x-1}$.

9. Graph the following relations in the given interval:

a. $y = |x| + x + 1$ for $-2 \le x \le 2$ [Hint: Sketch by adding ordinates]

b. $y = |x| + |x-1|$ for $-2 \le x \le 3$

c. $y = 2^x + 2^{-x}$ for $-2 \le x \le 2$

d. $|x - y| = 1$ for $-1 \le x \le 3$.

10. Sketch the function $f(x) = |x^2 - 1| - 1$.

11. Given $y = f(x)$ as sketched below, sketch

a. $y = 2f(x)$

b. $y = -f(x)$

c. $y = f(-x)$

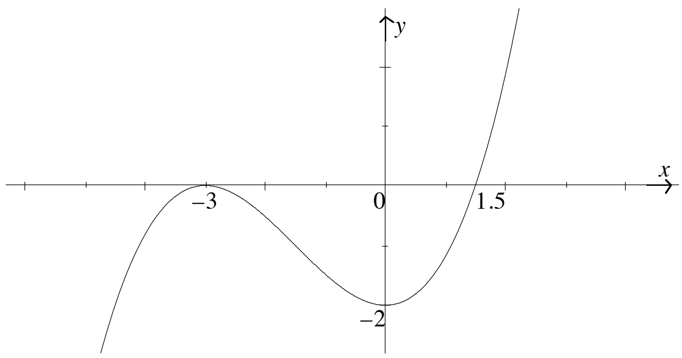
d. $y = f(x) + 4$

e. $y = f(x - 3)$

f. $y = f(x + 1) - 2$

g. $y = 3 - 2f(x - 3)$

h. $y = |f(x)|$



12. By sketching graphs solve the following equations:

a. $|2x| = 4$

b. $\frac{1}{x-2} = -1$

c. $x^3 = x^2$

d. $x^2 = \frac{1}{x}$

13. Solve $|x - 2| = 3$.

a. algebraically

b. geometrically.

14. The parabolas $y = (x - 1)^2$ and $y = (x - 3)^2$ intersect at a point P. Find the coordinates of P.

15. Sketch the circle $x^2 + y^2 - 2x - 14y + 25 = 0$. [Hint: Complete the squares.] Find the values of k, so that the line $y = k$ intersects the circle in two distinct points.

16. Solve $\frac{4}{5-x} = 1$, using a graph.

17. Find all real numbers x for which $|x - 2| = |x + 2|$.

18. Given that $Q(p) = p^2 - p$, find possible values of n if $Q(n) = 2$.

19. Solve $|x - 4| = 2x$.

a. algebraically

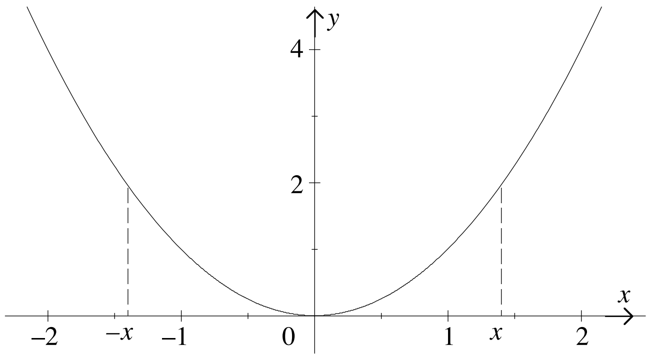
b. geometrically.

2.9 Even and odd functions

Definition:

> A function, $y = f(x)$, is *even* if $f(x) = f(-x)$ for all x in the domain of f.

Geometrically, an even function is symmetrical about the y-axis (it has line symmetry).
The function $f(x) = x^2$ is an even function as $f(-x) = (-x)^2 = x^2 = f(x)$ for all values of x. We illustrate this on the following graph.

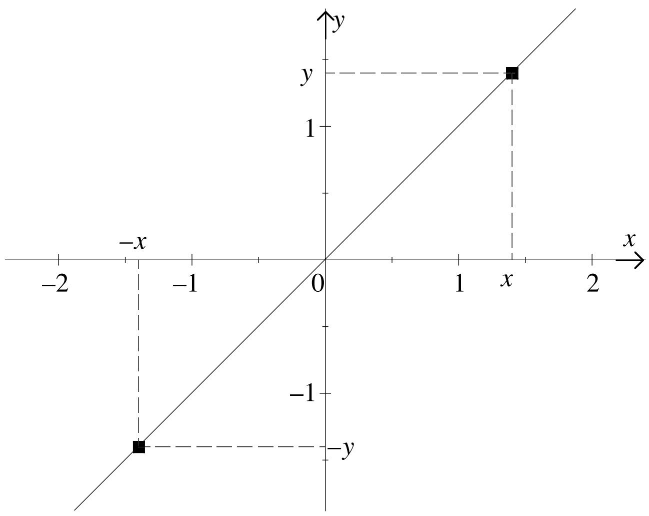


The graph of $y = x^2$.

Definition:

> A function, $y = f(x)$, is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Geometrically, an odd function is symmetrical about the origin (it has rotational symmetry).
The function $f(x) = x$ is an odd function as $f(-x) = -x = -f(x)$ for all values of x. This is illustrated on the following graph.



The graph of $y = x$.

Example

Decide whether the following functions are even, odd or neither.

1. $f(x) = 3x^2 - 4$

2. $g(x) = \frac{1}{2x}$

3. $f(x) = x^3 - x^2$.

Solution

1.

$$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$$

The function $f(x) = 3x^2 - 4$ is even.

2.

$$g(-x) = \frac{1}{2(-x)} = \frac{1}{-2x} = -\frac{1}{2x} = -g(x)$$

Therefore, the function g is odd.

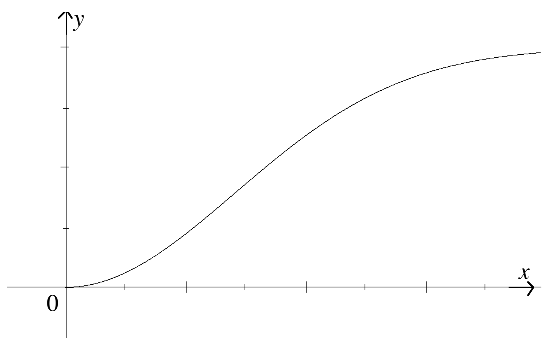
3.

$$f(-x) = (-x)^3 - (-x)^2 = -x^3 - x^2$$

This function is neither even (since $-x^3 - x^2 \neq x^3 - x^2$) nor odd (since $-x^3 - x^2 \neq -(x^3 - x^2)$).

Example

Sketched below is part of the graph of $y = f(x)$.

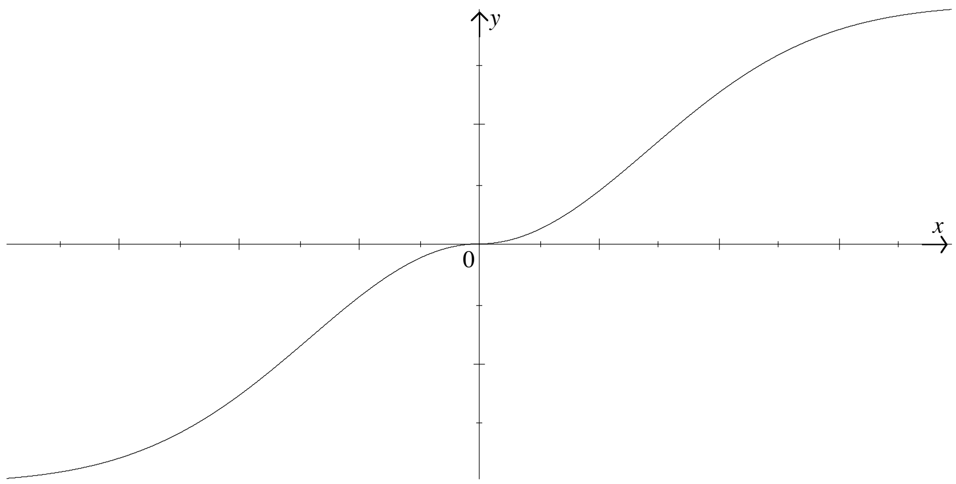


Complete the graph if $y = f(x)$ is

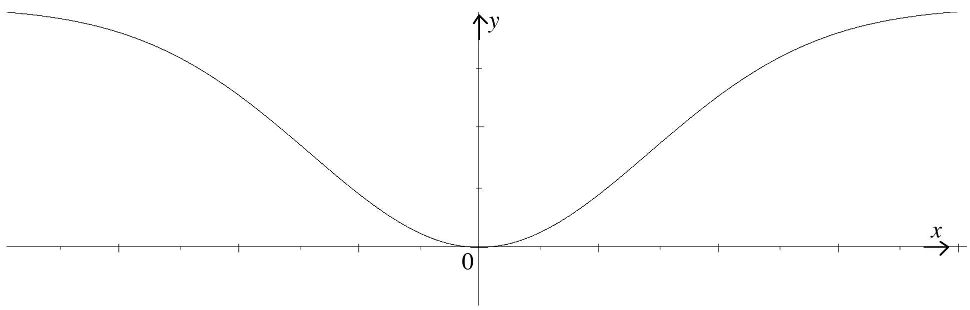
1. odd

2. even.

Solution



$y = f(x)$ is an odd function.



$y = f(x)$ is an even function.

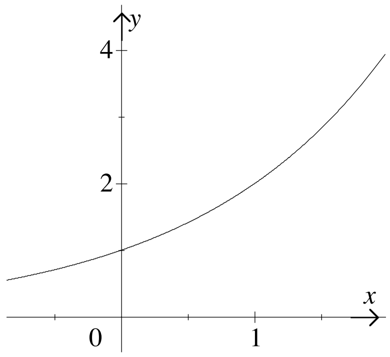
2.10 Increasing and decreasing functions

Here we will introduce the concepts of increasing and decreasing functions. In Chapter 5 we will relate these concepts to the derivative of a function.

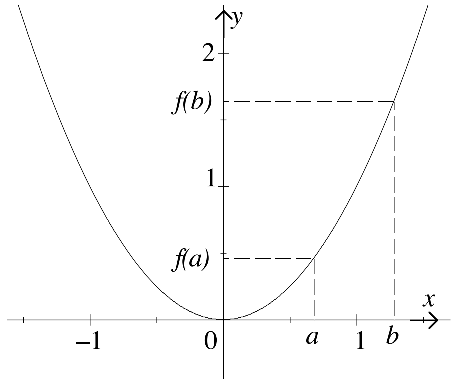
Definition:

> A function is *increasing* on an interval I, if for all a and b in I such that $a < b$, $f(a) < f(b)$.

The function $y = 2^x$ is an example of a function that is increasing over its domain. The function $y = x^2$ is increasing for all real $x > 0$.



The graph of $y = 2^x$. This function is increasing for all real x.



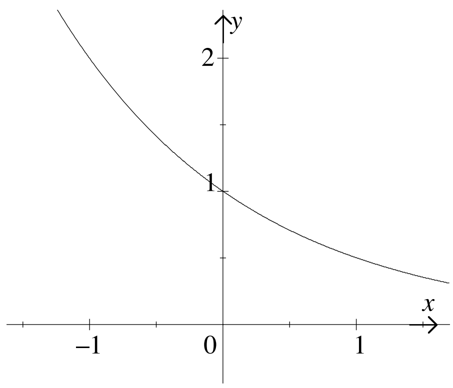
The graph of $y = x^2$. This function is increasing on the interval $x > 0$.

Notice that when a function is increasing it has a positive slope.

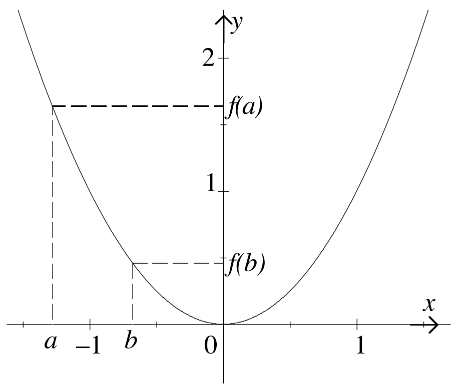
Definition:

> A graph is decreasing on an interval I, if for all a and b in I such that $a < b$, $f(a) > f(b)$.

The function $y = 2^{-x}$ is decreasing over its domain. The function $y = x^2$ is decreasing on the interval $x < 0$.



The graph of $y = 2^{-x}$. This function is decreasing for all real x.



The graph of $y = x^2$. This function is decreasing on the interval $x < 0$.

Notice that if a function is decreasing then it has negative slope.

2.11 Exercises

1. Given the graph below of $y = f(x)$:

a. State the domain and range.

b. Where is the graph

i. increasing?

ii. decreasing?

c. if k is a constant, find the values of k such that $f(x) = k$ has

i. no solutions

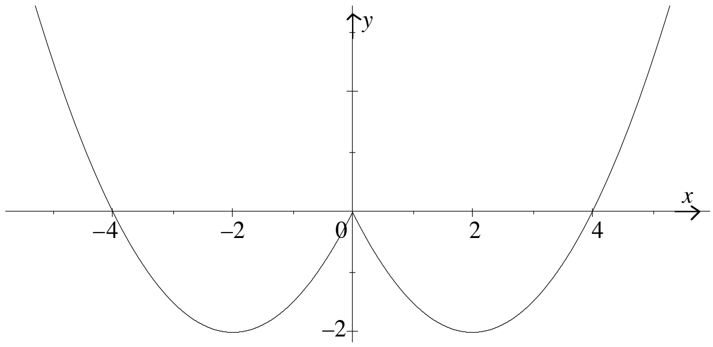
ii. 1 solution

iii. 2 solutions

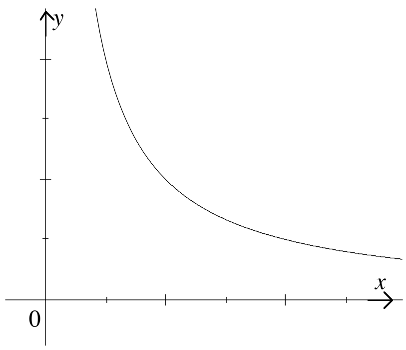
iv. 3 solutions

v. 4 solutions.

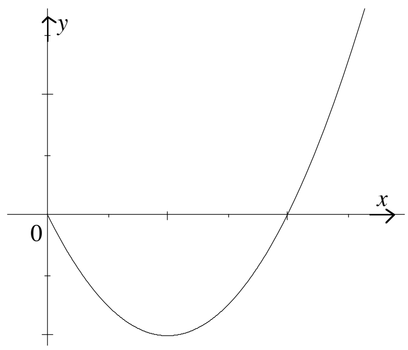
d. Is $y = f(x)$ even, odd or neither?



2. Complete the following functions if they are defined to be (a) even (b) odd.



$y = f(x)$



$y = g(x)$

3. Determine whether the following functions are odd, even or neither.

a. $y = x^4 + 2$ **b.** $y = \sqrt{4 - x^2}$ **c.** $y = 2^x$ **d.** $y = x^3 + 3x$

e. $y = \frac{x}{x^2}$ **f.** $y = \dfrac{1}{x^2 - 4}$ **g.** $y = \dfrac{1}{x^2 + 4}$ **h.** $y = \dfrac{x}{x^3 + 3}$

i. $y = 2^x + 2^{-x}$ **j.** $y = |x - 1| + |x + 1|$

4. Given $y = f(x)$ is even and $y = g(x)$ is odd, prove

a. if $h(x) = f(x) \cdot g(x)$ then $h(x)$ is odd

b. if $h(x) = (g(x))^2$ then $h(x)$ is even

c. if $h(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, then $h(x)$ is odd

d. if $h(x) = f(x) \cdot (g(x))^2$ then $h(x)$ is even.

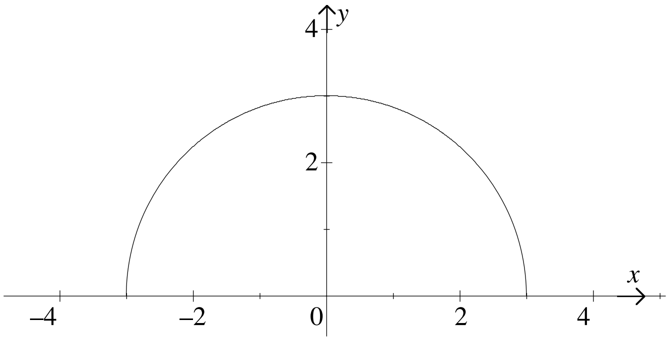
5. Consider the set of all odd functions which are defined at $x = 0$. Can you prove that for every odd function in this set $f(0) = 0$? If not, give a counter-example.

5 Solutions to exercises

1.4 Solutions

1. **a.** The domain of $f(x) = \sqrt{9 - x^2}$ is all real x where $-3 \le x \le 3$. The range is all real y such that $0 \le y \le 3$.

b.

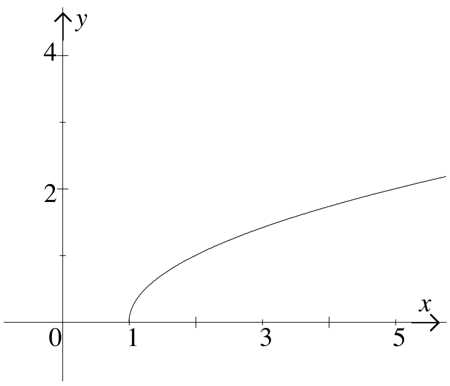


The graph of $f(x) = \sqrt{9 - x^2}$.

2.

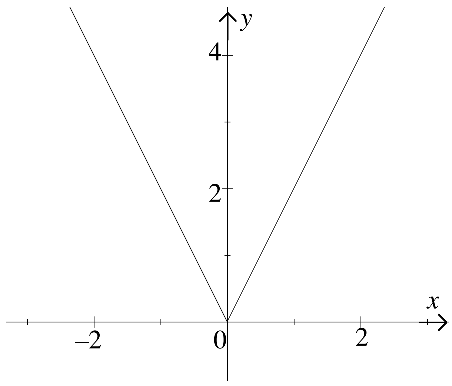
$$\begin{aligned}
\frac{\psi(x+h) - \psi(x)}{h} &= \frac{(x+h)^2 + 5 - (x^2 + 5)}{h} \\
&= \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h} \\
&= \frac{h^2 + 2xh}{h} \\
&= h + 2x
\end{aligned}$$

3. **a.**



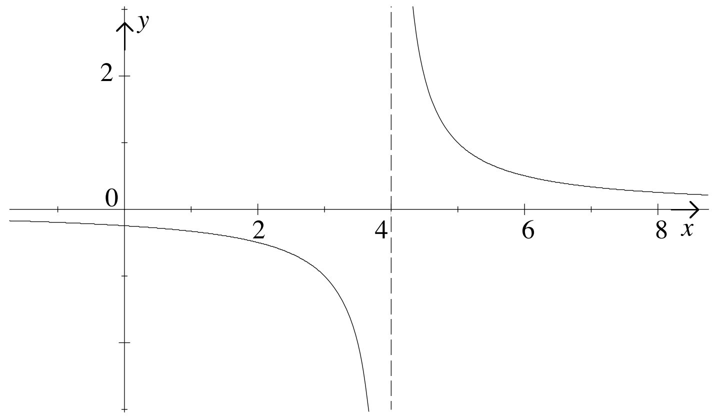
The graph of $y = \sqrt{x - 1}$. The domain is all real $x \ge 1$ and the range is all real $y \ge 0$.

b.



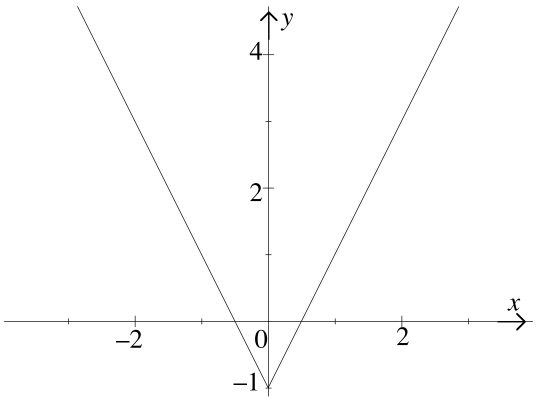
The graph of $y = |2x|$. Its domain is all real x and range all real $y \ge 0$.

c.



The graph of $y = \frac{1}{x-4}$. The domain is all real $x \neq 4$ and the range is all real $y \neq 0$.

d.

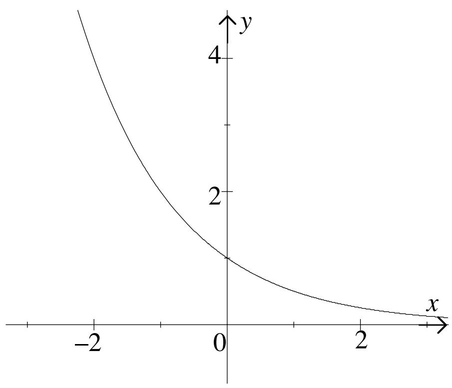


The graph of $y = |2x| - 1$. The domain is all real x, and the range is all real $y \geq -1$.

4. a. The perpendicular distance d from $(0,0)$ to $x + y + k = 0$ is $d = |\frac{k}{\sqrt{2}}|$.

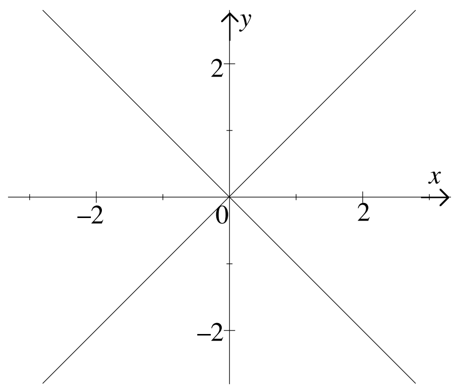
b. For the line $x+y+k = 0$ to cut the circle in two distinct points $d < 2$. ie $|k| < 2\sqrt{2}$ or $-2\sqrt{2} < k < 2\sqrt{2}$.

5. a.



The graph of $y = (\frac{1}{2})^x$.

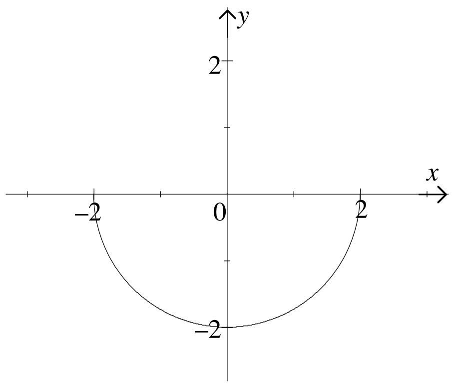
b.



The graph of $y^2 = x^2$.

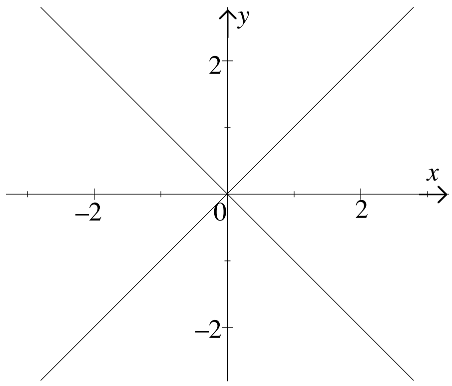
6. $y^2 = x^3$ is not a function.

7. a.



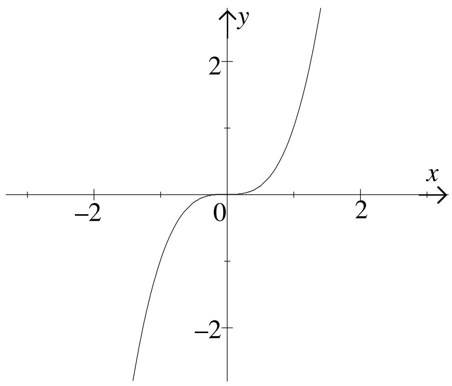
The graph of $y = -\sqrt{4-x^2}$. This is a function with the domain: all real x such that $-2 \le x \le 2$ and range: all real y such that $-2 \le y \le 0$.

b.



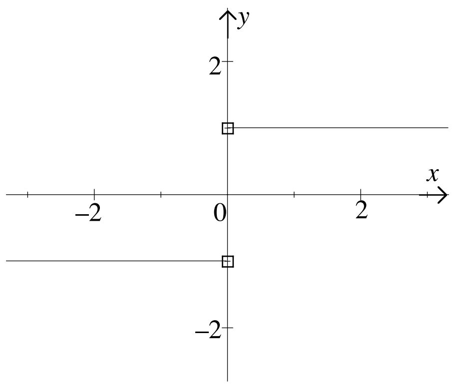
The graph of $|x| - |y| = 0$. This is not the graph of a function.

c.



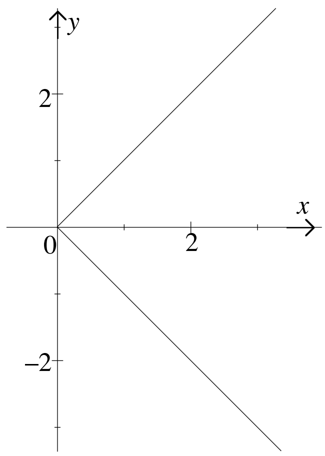
The graph of $y = x^3$. This is a function with the domain: all real x and range: all real y.

d.



The graph of $y = \frac{x}{|x|}$. This is the graph of a function which is not defined at $x = 0$. Its domain is all real $x \neq 0$, and range is $y = \pm 1$.

e.



The graph of $|y| = x$. This is not the graph of a function.

8.

$$\begin{aligned} A(\frac{1}{p}) &= (\frac{1}{p})^2 + 2 + \frac{1}{(\frac{1}{p})^2} \\ &= \frac{1}{p^2} + 2 + \frac{1}{\frac{1}{p^2}} \\ &= \frac{1}{p^2} + 2 + p^2 \\ &= A(p) \end{aligned}$$

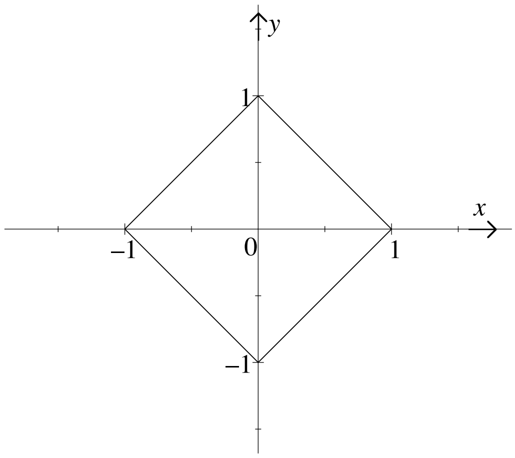
9. **a.** The values of x in the interval $0 < x < 4$ are not in the domain of the function.
b. $x = 1$ and $x = -1$ are not in the domain of the function.

10. **a.** $\phi(3) + \phi(4) + \phi(5) = \log(2.5)$
b. $\phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n) = \log(\frac{n}{2})$

11. **a.** $y = 3$ when $z = 3$.
b. **i.** $L(M(x)) = 2(x^2 - x) + 1$
ii. $M(L(x)) = 4x^2 + 2x$

12. **a.** $a = 2$, $b = 2$ so the equations is $y = 2x^2 - 2$.
b. $a = 5$, $b = 1$ so the equation is $y = \frac{5}{x^2+1}$.

13. **b.**



The graph of $|x| + |y| = 1$.

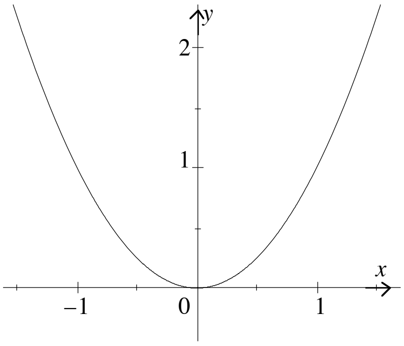
14. $S(n-1) = \frac{n-1}{2n-1}$

Hence

$$\begin{aligned} S(n) - S(n-1) &= \frac{n}{2n+1} - \frac{n-1}{2n-1} \\ &= \frac{n(2n-1) - (2n+1)(n-1)}{(2n-1)(2n+1)} \\ &= \frac{2n^2 - n - (2n^2 - n - 1)}{(2n-1)(2n+1)} \\ &= \frac{1}{(2n-1)(2n+1)} \end{aligned}$$

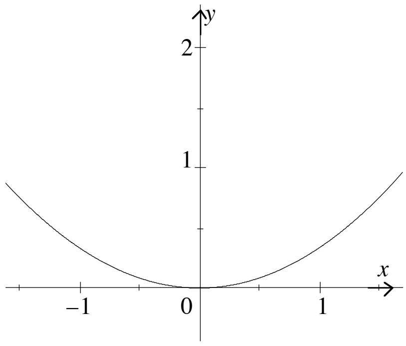
2.8 Solutions

1. a.



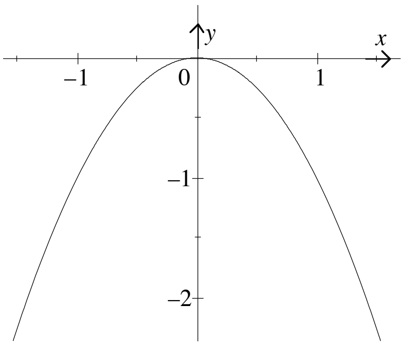
The graph of $y = x^2$.

b.



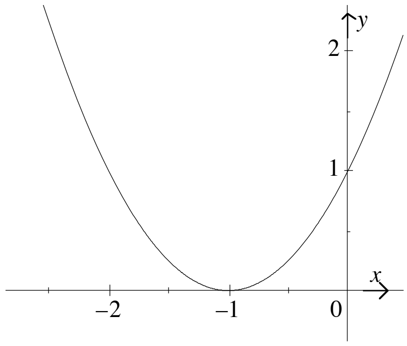
The graph of $y = \frac{x^2}{3}$.

c.



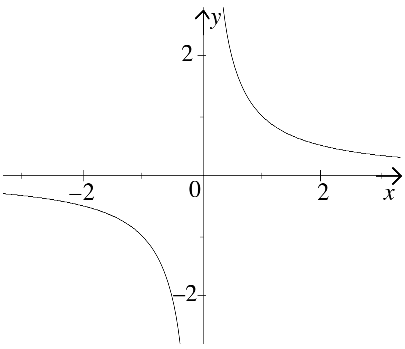
The graph of $y = -x^2$.

d.



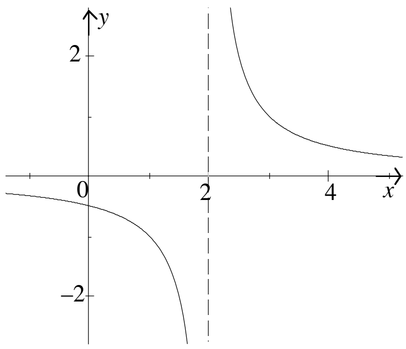
The graph of $y = (x + 1)^2$.

2. a.



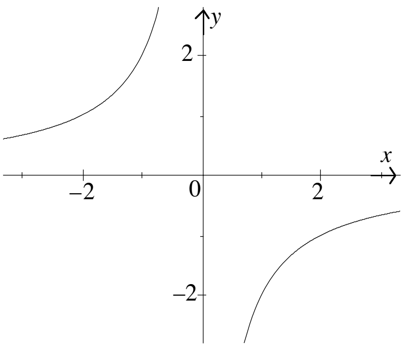
The graph of $y = \frac{1}{x}$.

b.



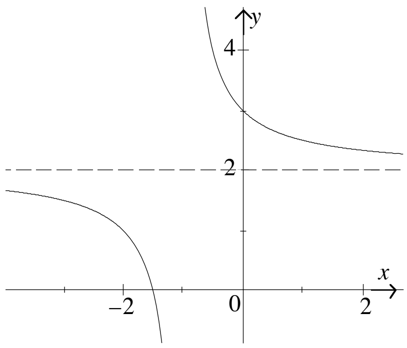
The graph of $y = \frac{1}{x-2}$.

c.



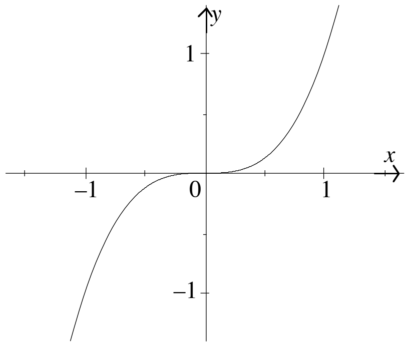
The graph of $y = \frac{-2}{x}$.

d.



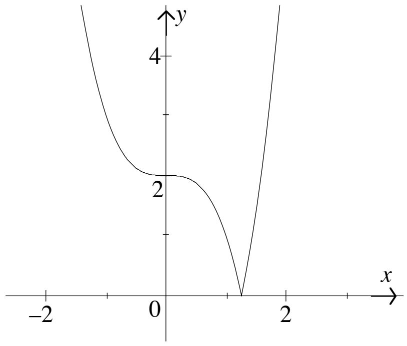
The graph of $y = \frac{1}{x+1} + 2$.

3. a.



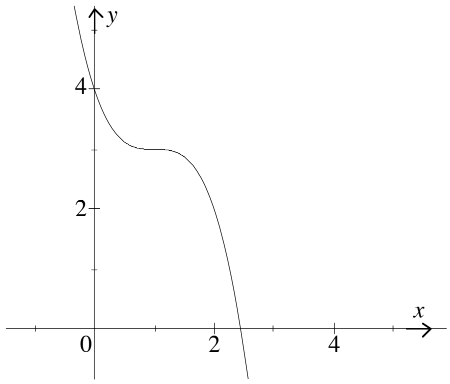
The graph of $y = x^3$.

b.



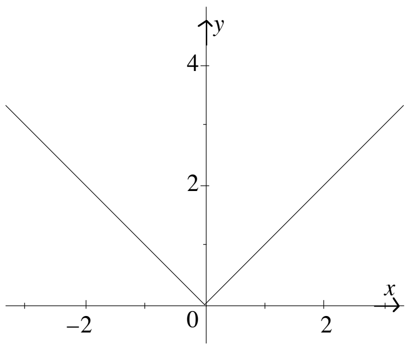
The graph of $y = |x^3 - 2|$.

c.



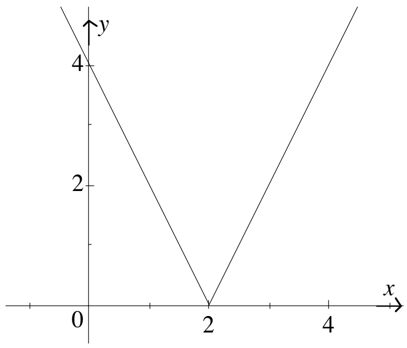
The graph of $y = 3 - (x - 1)^3$.

4. a.



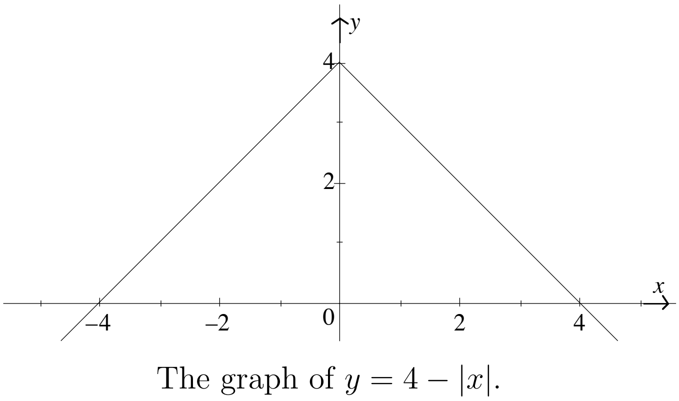
The graph of $y = |x|$.

b.



The graph of $y = 2|x - 2|$.

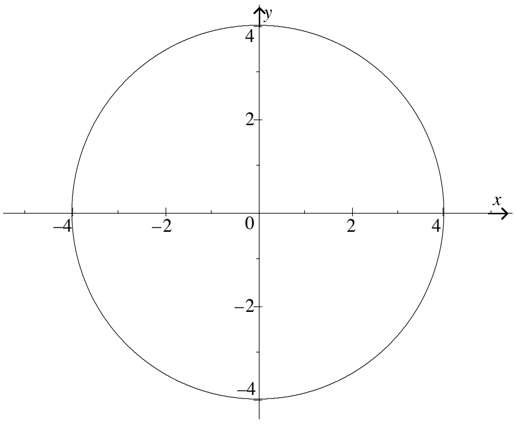
c.



The graph of $y = 4 - |x|$.

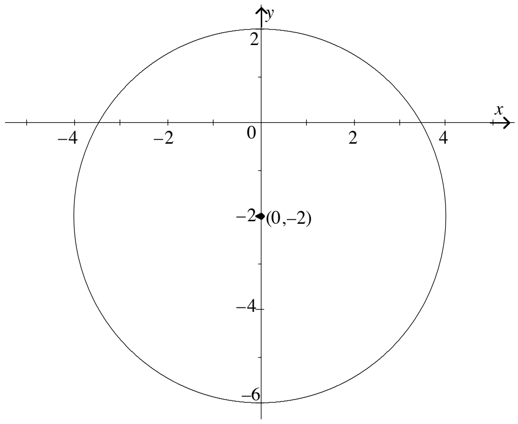
5.

a.



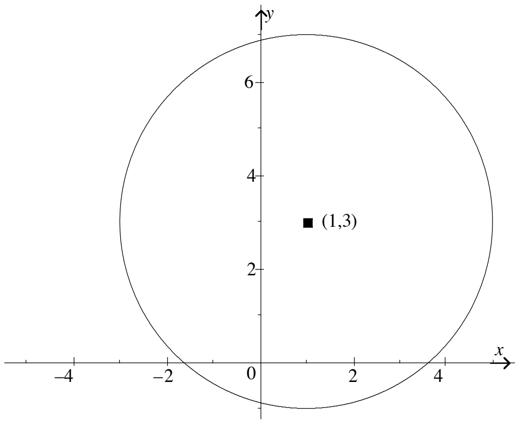
The graph of $x^2 + y^2 = 16$.

b.



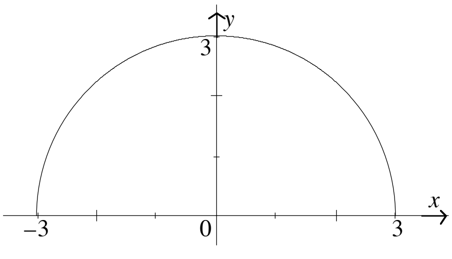
The graph of $x^2 + (y + 2)^2 = 16$.

c.



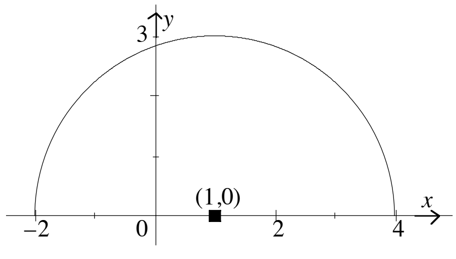
The graph of $(x - 1)^2 + (y - 3)^2 = 16$.

6. a.



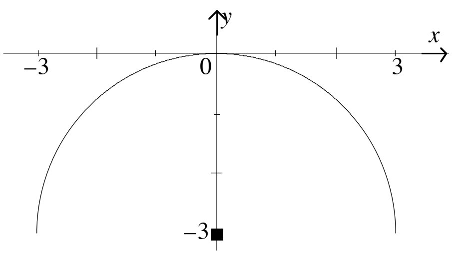
The graph of $y = \sqrt{9 - x^2}$.

b.



The graph of $y = \sqrt{9 - (x - 1)^2}$.

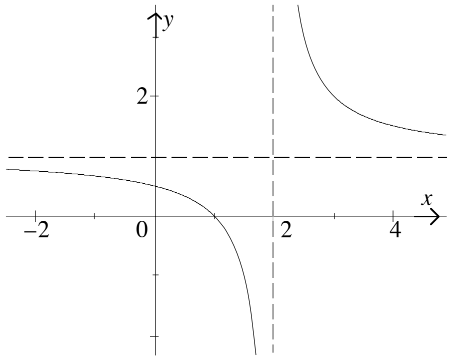
c.



The graph of $y = \sqrt{9 - x^2} - 3$.

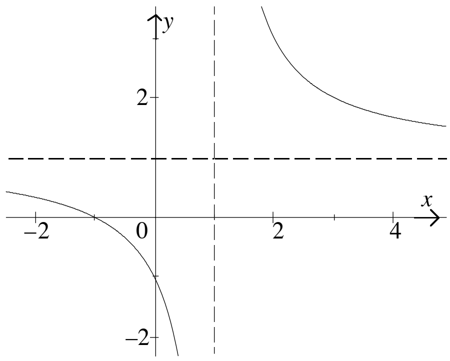
7.

$$\frac{1}{x-2}+1=\frac{1+(x-2)}{x-2}=\frac{x-1}{x-2}$$



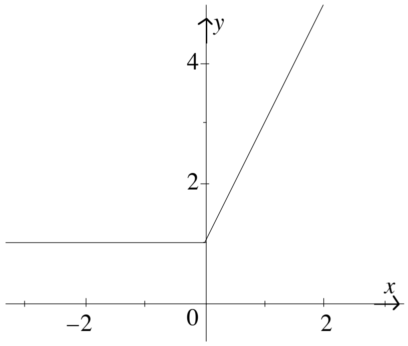
The graph of $y=\frac{x-1}{x-2}$.

8.



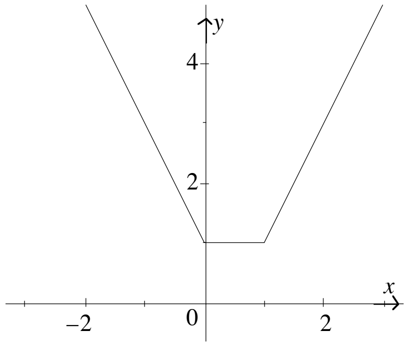
The graph of $y=\frac{x+1}{x-1}$.

9. a.



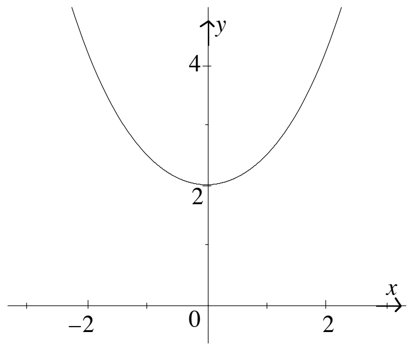
The graph of $y=|x|+x+1$ for $-2\le x\le 2$.

b.



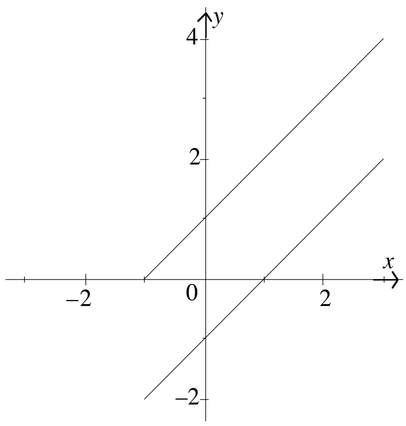
The graph of $y=|x|+|x-1|$ for $-2\le x\le 3$.

c.



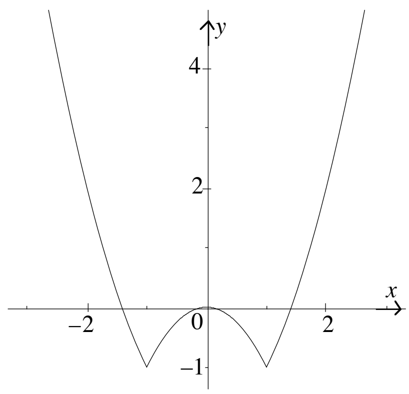
The graph of $y = 2^x + 2^{-x}$ for $-2 \leq x \leq 2$.

d.



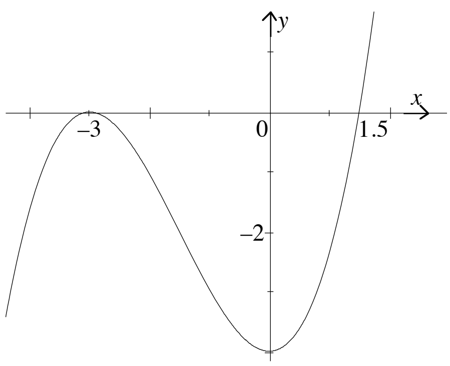
The graph of $|x - y| = 1$ for $-1 \leq x \leq 3$.

10.



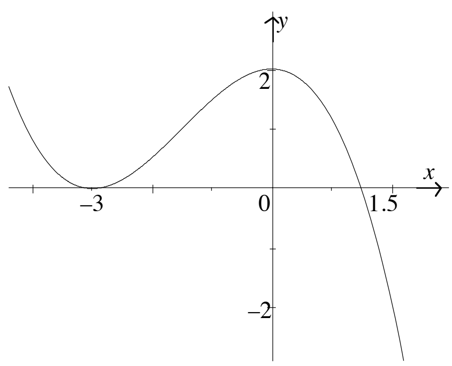
The graph of $f(x) = |x^2 - 1| - 1$.

11. a.



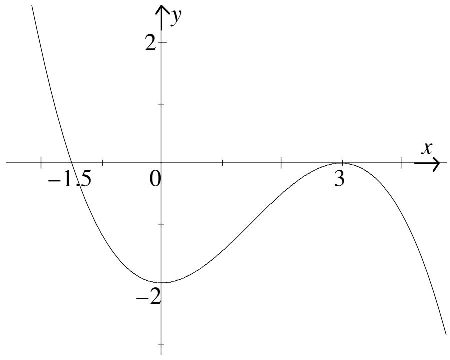
The graph of $y = 2f(x)$.

b.



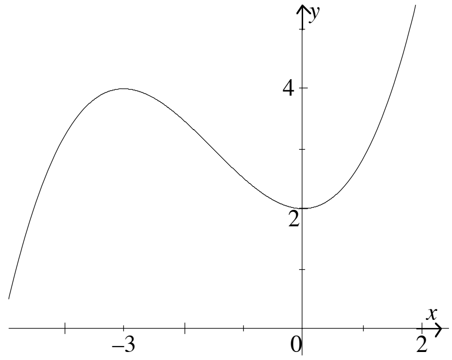
The graph of $y = -f(x)$.

c.



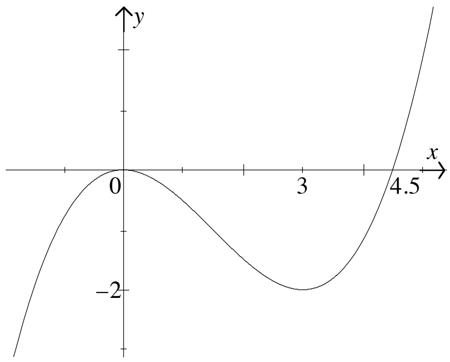
The graph of $y = f(-x)$.

d.



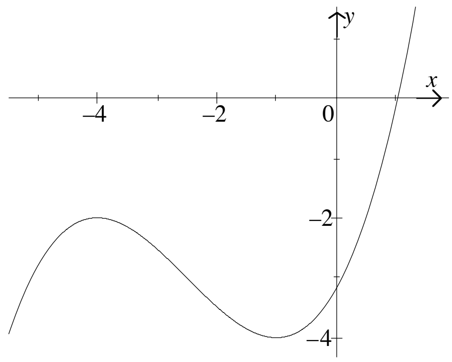
The graph of $y = f(x) + 4$.

e.



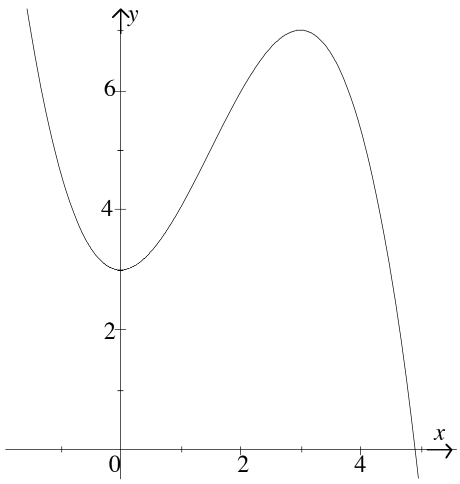
The graph of $y = f(x - 3)$.

f.



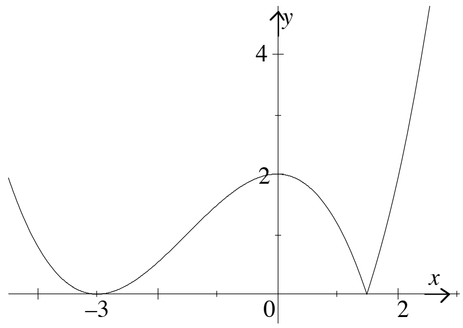
The graph of $y = f(x + 1) - 2$.

g.



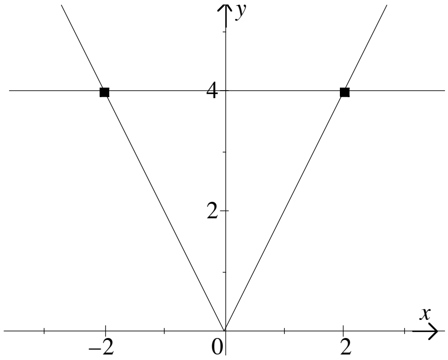
The graph of $y = 3 - 2f(x - 3)$.

h.



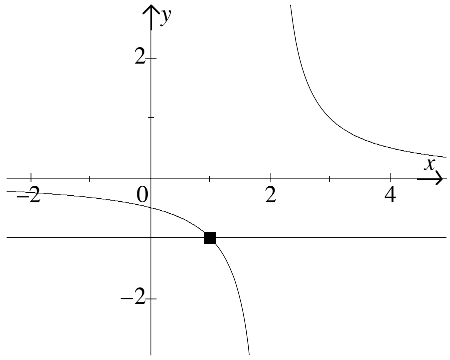
The graph of $y = |f(x)|$.

12. a.



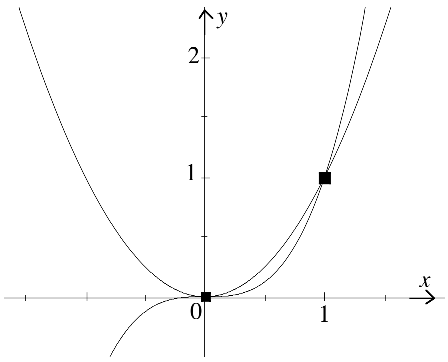
$x = -2$ and $x = 2$ are solutions of the equation $|2x| = 4$.

b.



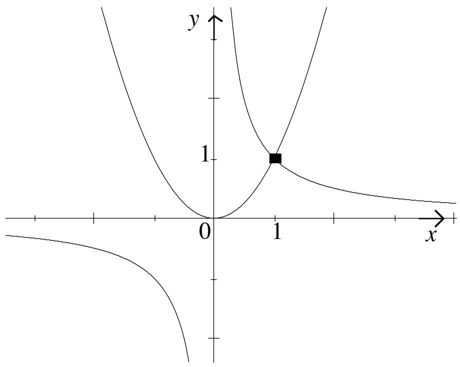
$x = 1$ is a solution of $\frac{1}{x-2} = -1$.

c.



$x = 0$ and $x = 1$ are solutions of the equation $x^3 = x^2$.

d.



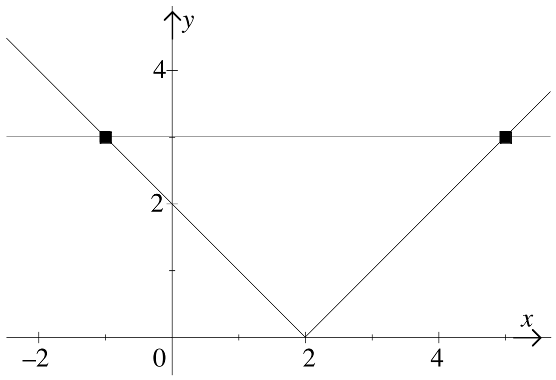
$x = 1$ is a solution of $x^2 = \frac{1}{x}$.

13.

a. For $x \geq 2$, $|x-2| = x-2 = 3$. Therefore $x = 5$ is a solution of the inequality. (Note that $x = 5$ is indeed ≥ 2.)

For $x < 2$, $|x-2| = -(x-2) = -x+2 = 3$. Therefore $x = -1$ is a solution. (Note that $x = -1$ is < 2.)

b.

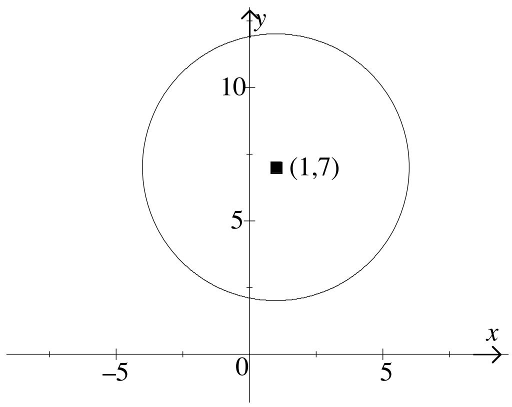


The points of intersection are $(-1, 3)$ and $(5, 3)$.

Therefore the solutions of $|x - 2| = 3$ are $x = -1$ and $x = 5$.

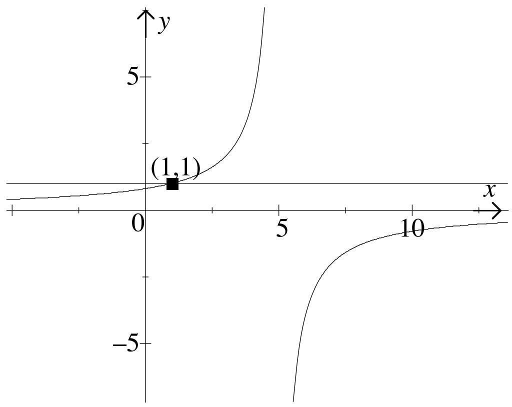
14. The parabolas intersect at $(2, 1)$.

15.



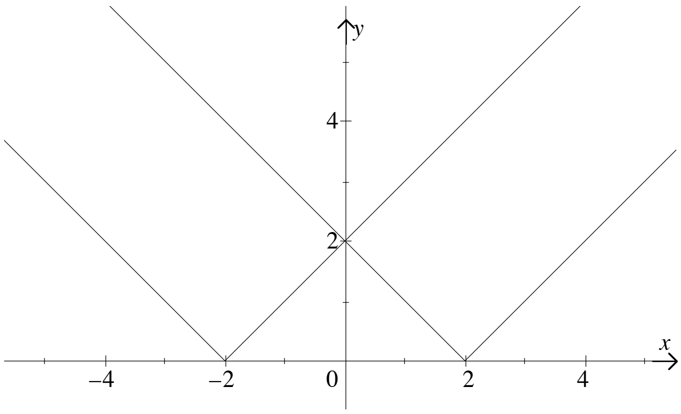
$y = k$ intersects the circle at two distinct points when $2 < k < 12$.

16.



The point of intersection is $(1, 1)$. Therefore the solution of $\frac{4}{5-x} = 1$ is $x = 1$.

17.

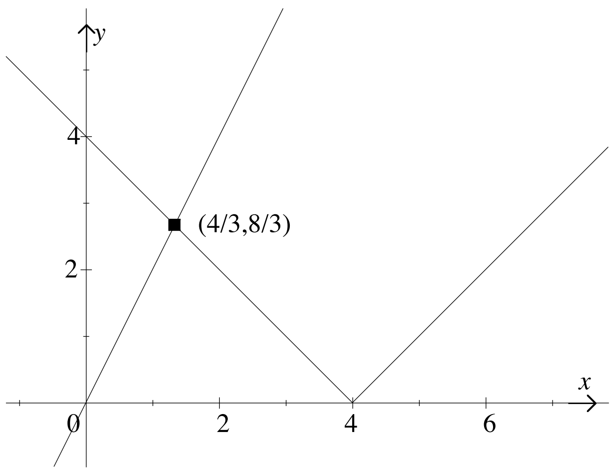


The point of intersection is $(0, 2)$. Therefore the solution of $|x-2| = |x+2|$ is $x = 0$.

18. $n = -1$ or $n = 2$.

19. a. For $x \geq 4$, $|x-4| = x-4 = 2x$ when $x = -4$, but this does not satisfy the condition of $x \geq 4$ so is not a solution.
For $x < 4$, $|x-4| = -x+4 = 2x$ when $x = \frac{4}{3}$. $x = \frac{4}{3}$ is < 4 so is a solution.
Therefore, $x = \frac{4}{3}$ is a solution of $|x-4| = 2x$.

b.



The graph of $y = |x-4|$ and $y = 2x$ intersect at the point $(\frac{4}{3}, \frac{8}{3})$. So the solution of $|x-4| = 2x$ is $x = \frac{4}{3}$.

2.11 Solutions

1. a. The domain is all real x, and the range is all real $y \geq -2$.

b. i. $-2 < x < 0$ or $x > 2$

ii. $x < -2$ or $0 < x < 2$

c. i. $k < -2$

ii. There is no value of k for which $f(x) = k$ has exactly one solution.

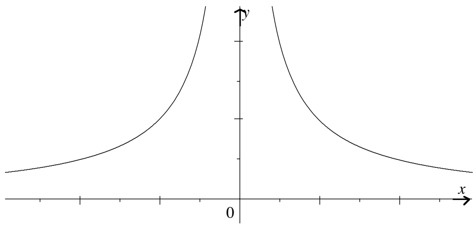
iii. $k = 2$ or $k > 0$

iv. $k = 0$

v. $-2 < k < 0$

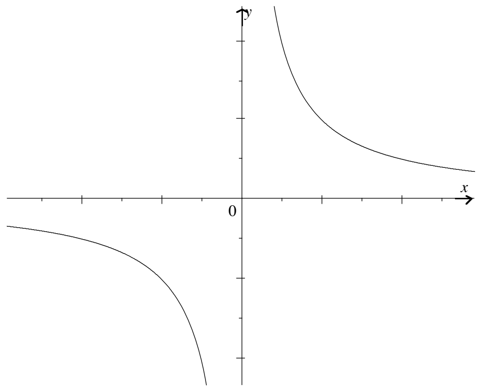
d. $y = f(x)$ is even

2. a.



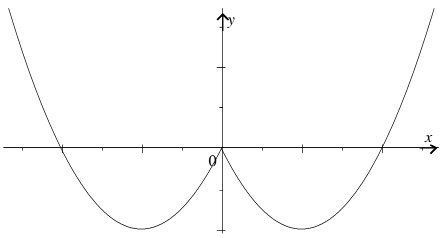
$y = f(x)$ is even.

b.



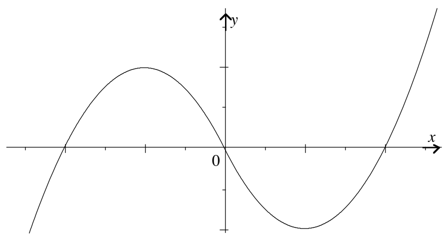
$y = f(x)$ is odd.

a.



$y = g(x)$ is even.

b.



$y = g(x)$ is odd.

3. **a.** even **b.** even **c.** neither **d.** odd **e.** odd

f. even **g.** even **h.** neither **i.** even **j.** even

4. a.

$$\begin{aligned} h(-x) &= f(-x) \cdot g(-x) \\ &= f(x) \cdot -g(x) \\ &= -f(x) \cdot g(x) \\ &= -h(x) \end{aligned}$$

Therefore h is odd.

b.

$$\begin{aligned} h(-x) &= (g(-x))^2 \\ &= (-(g(x))^2 \\ &= (g(x))^2 \\ &= h(x) \end{aligned}$$

Therefore h is even.

c.

$$\begin{aligned} h(-x) &= \frac{f(-x)}{g(-x)} \\ &= \frac{f(x)}{-g(x)} \\ &= -\frac{f(x)}{g(x)} \\ &= -h(x) \end{aligned}$$

Therefore h is odd.

d.

$$\begin{aligned} h(-x) &= f(-x)\cdot(g(-x))^2 \\ &= f(x)\cdot(-g(x))^2 \\ &= f(x)\cdot(g(x))^2 \\ &= h(x) \end{aligned}$$

Therefore h is even.

5. If f is defined at $x = 0$

$$\begin{aligned} f(0) &= f(-0) && \text{(since } 0 = -0\text{)} \\ &= -f(0) && \text{(since } f \text{ is odd)} \\ 2f(0) &= 0 && \text{(adding } f(0) \text{ to both sides)} \\ \text{Therefore} \quad f(0) &= 0. \end{aligned}$$

3.2 Solutions

1. **a.** $2f(-1) + f(2) = 2(1-(-1)) + (1-(2)^2) = 4 + (-3) = 1.$

b. $f(a^2) = 1 - (a^2)^2 = 1 - a^4$ since $a^2 \geq 0$.

2. You can see from the graph below that there is one solution to $f(x) = 2$, and that this solution is at $x = -1$.