Qualitative study of functions (exercises with detailed solutions)

Exercise 1 *Let* $f(x) = e^{2x} - 3e^x + 2$.

a) *Find the domain, the limits at the endpoints of the domain and the asymptotes. Find for which values of* x f *vanishes and study the sign of* f.

b) *Find the monotonicity intervals, local and global minima and maxima of* f.

c) *Find the convexity and concavity intervals and the inflection points of* f.

d) *Draw a qualitative graph of* f.

e) *Discuss the existence of solutions for the equation* $e^{2x} - 3e^x = \alpha$ *where* $\alpha \in \mathbb{R}$.

$\mathrm{dom}(f) = \mathbb{R}$ and $f(x) = (e^x - 1)(e^x - 2)$. Hence f vanishes when $x = 0$ and when $x = \log 2$. $f > 0$ if $x < 0$ or $x > \log 2$; $f < 0$ if $0 < x < \log 2$. We have

$$\lim_{x\to-\infty} f(x) = 2, \qquad \lim_{x\to+\infty} f(x) = +\infty.$$

$y = 2$ is an horiz. asymp. as $x \to -\infty$; As $x \to +\infty$, f does not have any asymptote.

The first derivative of f is

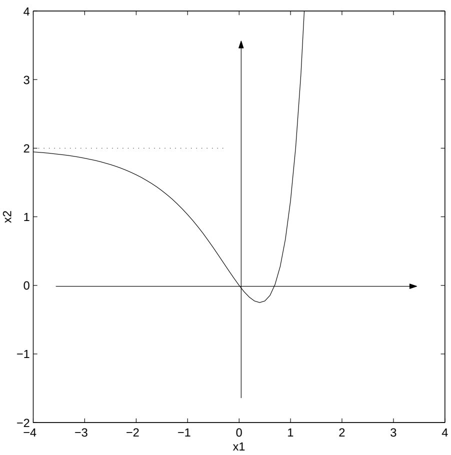
$$f'(x) = 2e^{2x} - 3e^x = e^x(2e^x - 3)$$

and it vanishes when $x = \log(3/2)$, is negative when $x < \log(3/2)$ (where f decreases) and positive when $x > \log(3/2)$ (where f increases). Since f is continuous we deduce that $x = \log\frac{3}{2}$ is a minimum.

The second derivative of f is

$$f''(x) = 4e^{2x} - 3e^x = e^x(4e^x - 3),$$

hence $x = \log(3/4)$ is an inflection point. f is concave if $x < \log(3/4)$, convex if $x > \log(3/4)$.



The minimum point we have found is an absolute minimum, while f does not have any maximum point (f is not bounded from above).

In order to answer to the last question, we remark that the minimal value of f is $-1/4$. Let $\beta = 2 + \alpha$, we have:

- the equation has no solutions if $\beta < -\frac{1}{4}$ (that is if $\alpha < -\frac{9}{4}$),
- the equation has two solutions if $-\frac{1}{4} < \beta < 2$ ($-\frac{9}{4} < \alpha < 0$),
- the equation has one solution if $\beta \geq 2$ or if $\beta = -\frac{1}{4}$ ($\alpha \geq 0$ or $\alpha = -\frac{9}{4}$.

Exercise 2 *Let* $f(x) = x + \log(x^2 - 5x + 6)$.

a) Find the domain, the limits at the endpoints of the domain and the asymptotes.

b) Find the monotonicity intervals, local and global minima and maxima of f.

c) Find the convexity and concavity intervals and the inflection points of f.

d) Draw a qualitative graph of f.

The function f is defined when $x \in I \cup J$, where $I = (-\infty, 2)$ and $J = (3, +\infty)$. Furthermore,

$$\lim_{x\to-\infty} f(x) = -\infty, \qquad \lim_{x\to 2^-} f(x) = -\infty, \qquad \lim_{x\to 3^+} f(x) = -\infty, \qquad \lim_{x\to+\infty} f(x) = +\infty$$

indeed just in the first limit (in the others we simply replace) we have an indeterminate form and we can solve it as follows

$$\lim_{x\to-\infty} x + \log(x^2 - 5x + 6) = \lim_{x\to-\infty} x + \log x^2 = \lim_{x\to-\infty} x + 2\log x = \lim_{x\to-\infty} x(1 + 2(\log x)/x) = -\infty.$$

f does not have any oblique asymptote.

We have

$$f'(x) = 1 + \frac{2x - 5}{x^2 - 5x + 6} = \frac{x^2 - 3x + 1}{x^2 - 5x + 6}, \qquad \forall x \in I \cup J.$$

$x^2 - 3x + 1 = 0$ has 2 solutions, $x = (3 \pm \sqrt{5})/2$, but only $x = (3 - \sqrt{5})/2 \in \mathrm{dom}(f)$ Furthermore

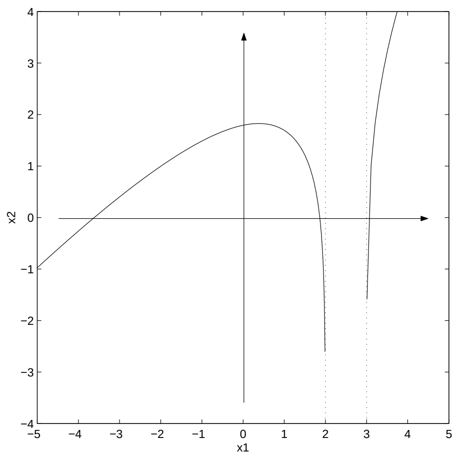
$$f'(x) > 0 \quad \text{if} \quad x < \frac{3 - \sqrt{5}}{2} \quad \text{or} \quad x > 3; \qquad\qquad f'(x) < 0 \quad \text{if} \quad \frac{3 - \sqrt{5}}{2} < x < 2.$$

Hence f has a (local) maximum at $x = (3 - \sqrt{5})/2$. The second derivative is

$$f''(x) = \frac{-2x^2 + 10x - 13}{(x^2 - 5x + 6)^2}.$$

$-2x^2 + 10x - 13 < 0$ for every $x \in \mathbb{R}$. Then f does not have any inflection points and it is concave both on I and on J (but not on $I \cup J$!).

In order to draw the graph of f, we study the intersections of its graph with the x-axes. At the endpoints of J we know that f tends to $-\infty$ and to $+\infty$ respectively, and that f increases on the whole interval. Then the graph has exactly one intersection with the x-axes in J.
On I, since the limits at the endpoints are both $-\infty$ we want to understand if $f(x) > 0$ for some $x \in I$.
The level of the maximum point is difficult to compute, but $f(0) = \log 6 > 0$. Since f is continuous and has a unique maximum we deduce the existence of exactly two intersections between its graph and the x-axes in the interval I.



Exercise 3 *Let* $f(x) = \sqrt{1 + \log(2 - x^2)}$.

a) *Find the domain of* f.

b) *Find the monotonicity intervals, local and global minima and maxima of* f.

c) *Draw a qualitative graph of* f.

d) *prove that* f *is invertible on* $\mathrm{dom}(f) \cap (-\infty, -1)$*, find* f^{-1} *specifying its domain and range.*

$\mathrm{dom}(f) = \left[-\sqrt{2 - e^{-1}}, \sqrt{2 - e^{-1}}\right]$, f is even and continuous on $\mathrm{dom}(f)$ (composition of continuous functions). Since $\mathrm{dom}(f)$ is closed and bounded, from Weierstrass' Theorem we deduce that f achieves its global maximum and minimum.
f is differentiable in $\left(-\sqrt{2 - e^{-1}}, \sqrt{2 - e^{-1}}\right)$ and

$$f'(x) = \frac{x}{(x^2 - 2)\sqrt{1 + \log(2 - x^2)}}.$$

$f'(x) = 0$ if and only if $x = 0$. Furthermore since

$$f'(x) > 0 \iff -\sqrt{2 - e^{-1}} < x < 0.$$

$x = 0$ is a maximum point for f. f increases in $(-\sqrt{2 - e^{-1}}, 0)$, decreases in $(0, \sqrt{2 - e^{-1}})$, hence at $x = 0$ f achieves its global maximum. The global minimum is achieved at two different points: $x = \pm\sqrt{2 - e^{-1}}$. We remark that at the endpoints of its domain f is not differentiable, indeed

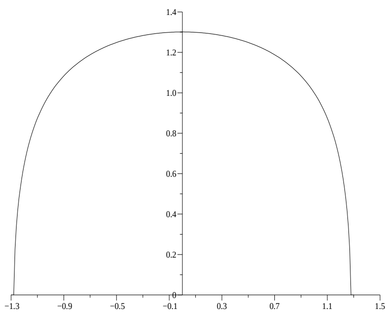
$$\lim_{x \to \left(\sqrt{2 - e^{-1}}\right)^-} f'(x) = -\infty, \qquad \lim_{x \to \left(-\sqrt{2 - e^{-1}}\right)^+} f'(x) = +\infty.$$

In order to draw the graph of f we remark that f vanishes when $x = \pm\sqrt{2 - e^{-1}}$ and that $f(0) = \sqrt{1 + \log 2}$. If we call g the restriction of f to $\left[-\sqrt{2 - e^{-1}}, -1\right)$ we have that g is injective, since strictly increasing. We can then invert g; since

$$\lim_{x \to -1} g(x) = 1 \quad \text{we have} \quad \mathcal{R}(g) = [0, 1).$$

Then

$$\mathrm{dom}(g^{-1}) = [0, 1), \qquad \mathcal{R}\left(g^{-1}\right) = \left[-\sqrt{2 - e^{-1}}, -1\right).$$



We now explicit x as a function of y in $y = g(x)$, that is

$$x = -\sqrt{2 - e^{y^2 - 1}}.$$

The inverse function is then

$$g^{-1}(x) = -\sqrt{2 - e^{x^2 - 1}}.$$

Exercise 4 *Let* $f(x) = \begin{cases} \dfrac{5 + 2\log|x|}{2 + \log|x|} & \textit{if } x \neq 0 \\ 2 & \textit{if } x = 0. \end{cases}$

a) *Find the domain, the limits at the endpoints of the domain and the asymptotes.*

b) *Find the monotonicity intervals, local and global minima and maxima of f.*

c) *Find the convexity and concavity intervals and the inflection points of f.*

d) *Draw a qualitative graph of f.*

$\mathrm{dom}(f) = \left(-\infty, -e^{-2}\right) \cup \left(-e^{-2}, e^{-2}\right) \cup \left(e^{-2}, +\infty\right)$ and f is even; then we just study it when $x \geq 0$. When $x > 0$ we have

$$f(x) = \frac{5 + 2\log x}{2 + \log x}.$$

$f(x) = 0$ when $x = e^{-\frac{5}{2}}$. $f(x) > 0$ when $0 \leq x < e^{-\frac{5}{2}}$ or $x > e^{-2}$, $f(x) < 0$ when $e^{-\frac{5}{2}} < x < e^{-2}$. The limits of f are

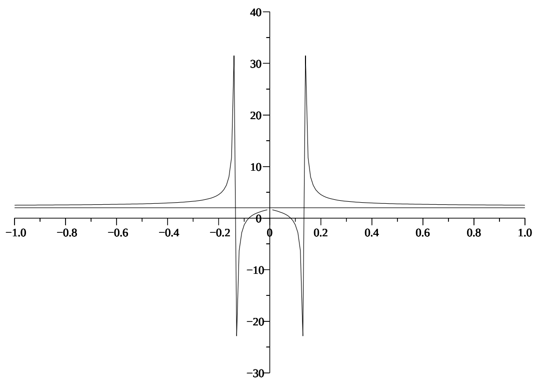
$$\lim_{x \to +\infty} f(x) = 2 \quad \Longrightarrow \quad y = 2 \text{ is an horiz. asymp.,}$$

$$\lim_{x \to (e^{-2})^{\pm}} f(x) = \pm\infty \quad \Longrightarrow \quad x = e^{-2} \text{ is a vert. asymp..}$$

Furthermore

$$\lim_{x \to 0^+} f(x) = 2$$

and f is continuous from the right at 0: since f is even we c
that it is continuous at 0. Then f is continuous on its dom
f is differentiable whenever $x > 0$ ($x \neq e^{-2}$), and

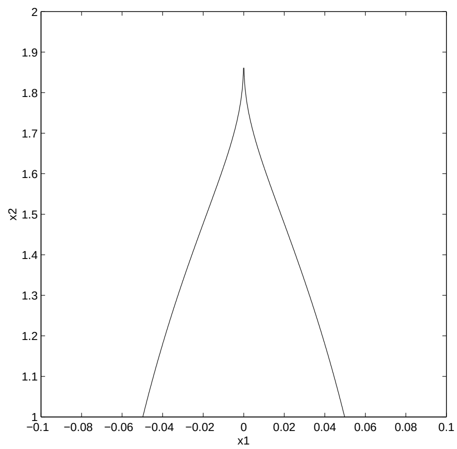


$$f'(x) = -\frac{1}{x(2 + \log x)^2}.$$

f is not differentiable at $x = 0$ (cusp), indeed

$$\lim_{x \to 0^+} f'(x) = -\infty, \qquad \lim_{x \to 0^-} f'(x) = +\infty.$$

$f'(x) \neq 0$ hence the extremal points belong to the set of
where f is not differentiable. Hence the unique (possible) extremal point is $x = 0$. $f'(x) > 0$ for every $x > 0$ ($x \neq e^{-2}$). and f increases in $\left(0, e^{-2}\right)$ and in $\left(e^{-2}, +\infty\right)$. $x = 0$ is a local maximum. f' is differentiable whenever $x > 0$ ($x \neq e^{-2}$) and



$$f''(x) = \frac{\log x + 4}{x^2(2 + \log x)^3}.$$

$f''(x) = 0$ when $x = e^{-4}$ and we have

$$f''(x) > 0 \quad \Longleftrightarrow \quad 0 < x < e^{-4}, \quad x > e^{-2}.$$

f is convex in $(0, e^{-4})$ and in $(-e^{-2}, +\infty)$, f is concave in (e^{-4}, e^{-2}). $x = e^{-4}$ is an inflection point.