How To Compute Taylor Error via the Remainder Estimation Theorem

This document goes over the fundamentals of how to use the Remainder Estimation Theorem in order to estimate the approximation error from using a Taylor polynomial. This is commonly used when you're given an approximating polynomial, such as

$$(1+x)^{2/3} \approx 1 + \frac{2}{3}x \text{ when } x \approx 0$$

or

$$e^{4x} \approx 1 + 4x + 8x^2 \text{ when } x \approx 0$$

In these examples, a function $f(x)$ is provided with an approximation of degree n around a center a. In both cases, $a = 0$... you can tell this because the x in the polynomials should be viewed like $(x - 0)$. (Thus, these are MacLaurin polynomials too.) In the first one, $n = 1$, and in the second one, $n = 2$.

Understanding what the Theorem Says

We use the Taylor polynomial $P_n(x)$ to approximate $f(x)$ when $x \approx a$, and the Taylor error is $R_n(x)$ which measures how far off $P_n(x)$ is from $f(x)$. The precise statement of the theorem is

Theorem (Remainder Estimation Theorem)**.** *Suppose the $(n+1)$st derivative $f^{(n+1)}(x)$ exists for all x in some interval I containing a. For all x in I,*

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

where M is the maximum value of $|f^{(n+1)}|$ in the interval.

First, we remark that this is an *absolute bound* on the error. It doesn't find $R_n(x)$... it finds the absolute value $|R_n(x)|$. That's why you see absolutes around $|x - a|$, and it's why M needs to be the maximum of the **absolute value of the** $(n+1)$**st derivative!**

Another note is that everything on the right side uses $n + 1$, not n. You take $n + 1$ derivatives. You use an $(n + 1)$st power of $|x - a|$. You divide by the $(n + 1)$st factorial. **The remainder goes one order higher** than $P_n(x)$ does.

Lastly, this theorem **needs an interval for** x **to make sense**. You may be provided that information in a couple different ways. You may be given an interval, like $[0, 10^{-5}]$. You may be given an inequality on $|x - a|$, like $|x| < 1/5$, which you would turn into $-1/5 < x < 1/5$.

Now, let's look at the parts one by one:

- $(n + 1)!$: Pretty self-explanatory.

- $|x - a|^{n+1}$: Remember that $|x - a|$ represents the distance between x and a. Thus, $|x - a|$ is biggest when x is as far from a as possible. Sometimes you're told this bound automatically, like $|x| < 1/5$. If not, then you have to look at the interval, see which side is farther from a, and use that distance.

- M: This is usually the hardest part to find. First of all, you need to compute the $(n+1)$st derivative $f^{(n+1)}(x)$. You need to take its absolute value. Lastly, you need to look over your whole interval and see where the value comes out biggest.

  In most cases, the derivative is always rising or always falling (i.e. it's monotone). When it's always rising, the biggest x value from the interval yields the biggest M. When it's falling, the smallest x value yields the biggest M. If you're not sure which to use, *try putting in both ends and seeing which one makes the bigger value.*

  You want to think of M as being "pessimistic": we don't know which point we'll use in the derivative, so we assume the worst and go with the biggest possible derivative (i.e. the least accurate error).

One other remark to note: M and $|x-a|$ *do not have to use the same x value!* Their parts are computed independently of one another!

Two Examples

Let's demonstrate this procedure on the two examples shown at the beginning of the document.

<u>Example 1:</u> Find the error in approximating $(1+x)^{2/3} \approx 1 + (2/3)x$ when $|x| < 1/2$.

<u>Answer:</u> Here, $f(x) = (1+x)^{2/3}$, $a = 0$, and $n = 1$. We're also told $|x| < 1/2$, so $-1/2 < x < 1/2$. We want to know

$$|R_1(x)| \le \frac{M|x|^2}{2!} \text{ where } M \text{ is max of } |f''(x)|$$

1. First, let's find M. Compute the second derivative: I leave out those steps... you get $f''(x) = -2/9(1+x)^{-5/3}$. Therefore, its absolute value is

   $$|f''(x)| = \frac{2}{9}|1+x|^{-5/3} \text{ aka } \frac{(2/9)}{|1+x|^{5/3}}$$

   It's up to you if you want to write the negative exponent as something in the denominator. I did so, because it makes this next step easier to think about.

   We found our interval earlier: $-1/2 < x < 1/2$. Since $|f''(x)|$ has a positive power in the denominator, it is a *decreasing* function, so its biggest value occurs at the smallest x. Thus, let's use $x = -1/2$ to get M:

   $$M = \frac{(2/9)}{|1+(-1/2)|^{5/3}} \text{ aka } \frac{2}{9}\left(\frac{1}{2}\right)^{-5/3}$$

2. Next, we need a bound on $|x|^2$. We were already told $|x| < 1/2$, so $|x|^2 < (1/2)^2$.

3. Put this all together:

   $$\frac{M|x|^2}{2!} \le \boxed{\frac{(2/9)(1/2)^{-5/3} \cdot (1/2)^2}{2!}}$$

Example 2: Compute the error from the MacLaurin polynomial $P_2(x)$ for $f(x) = e^{4x}$ when x is in $[-1, 0.1]$.

Answer: Here, $a = 0$ (because MacLaurin always implies $a = 0$) and $n = 2$. We're told the interval is $[-1, 0.1]$. Note that this interval is *not symmetric about* $a = 0$: one end is 1 unit away and the other is 0.1 units away. We want

$$|R_2(x)| \leq \frac{M|x|^3}{3!} \text{ where } M \text{ is max of } |f'''(x)|$$

1. Let's find M. We need the third derivative for $R_2(x)$... I eventually get $f'''(x) = 4^3 e^{4x}$. The absolute value is

   $$|f'''(x)| = 4^3 e^{4x}$$

   Note that 4^3 and e^{4x} are already positive, so the absolute value didn't change them at all.

   Now, this is an increasing function! Thus, its biggest value occurs at the biggest x of my interval $[-1, 0.1]$. Thus, we'll plug in $x = 0.1$ here:

   $$M = 4^3 e^{4 \cdot 0.1} = 64e^{0.4}$$

2. We need a bound on $|x|^3$ in the formula. We were *not told* $|x|$ this time! This is when you have to ask yourself which end of the interval is farther from a.

   We found earlier that the left end is 1 unit away (i.e. $|(-1) - 0| = 1$) and the other is 0.1 units away. The bigger value is 1, so we say $|x| \leq 1$.

3. Put this together:

   $$\frac{M|x|^3}{3!} \leq \boxed{\frac{64e^{0.4} \cdot (1)^3}{3!}}$$

Remark: What if the interval were changed to $[0, 0.2]$ instead? What would change in the work and answer?

Answer: The third derivative would not change. We would still find the biggest value at the largest x. This time, that x value is 0.2, so we'd get $M = 4^3 e^{4 \cdot 0.2}$.

The bound on $|x|$ would definitely change! Now, our left end is 0 units away from a, and the right end is 0.2 units away from a. The bigger value of these is 0.2, so we would use $|x| \leq 0.2$.

Our error would become $\dfrac{64e^{0.8} \cdot (0.2)^3}{3!}$.